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Unified approach to acoustic and electromagnetic field theories based on control engineering methods

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***Abstract*—In this paper, we propose a unified approach to acoustic and electromagnetic field theories which employs control engineering methods for their analysis and modelling. Both theories can be derived from the wave equation using factorisation and subsequently represented as a system with a feedback loop in control engineering. This allows for the formulation of properties and solutions useful for further analysis. Moreover, it provides a justification and explanation of similarities between acoustics and electromagnetism. Hopefully, our unified approach to acoustic and electromagnetic field theories carries implications for the foundational understanding of both theories as well as their practical applications.**

***Keywords*—acoustic field theory; electromagnetic field theory; electromagnetic-acoustic analogy**

I. Introduction

ACOUSTIC waves are similar to electromagnetic waves in the sense that both are described by the wave equation. However, they also possess opposite properties because acoustic waves are a type of mechanical waves which require a medium for propagation, whereas electromagnetic waves can propagate in vacuum.

In [1], arguments are provided for treating the dynamic phenomena of acoustics and electromagnetism in the same way. That is, causality and locality are considered as the core of dynamic linear phenomena in the physical continua of acoustics and electromagnetism. Then, the author discusses similarities between both fields in terms of energy and momentum conservation. In [2], an approach to the fundamental problems of classical linear acoustics and electromagnetism is developed, based on similarities between both fields. The author notes the importance of causality as a consequence of feedback loops existing between complementary quantities occurring in both fields. Then, several types of excitations are introduced, for which analytical solutions are calculated. A new notation of space physical features and the related local dynamical states is proposed and applied in the four cases of time-domain fundamental solutions related to the given sources, stressing the mutual circular coupling of these states and exposing in each case their duality, hierarchy, and source-dependence. In [3], an integrated view on acoustics and electromagnetism is developed. It introduces the notion of wave bifields and develops the concept of kinedynamics. Then, the dynamics of the fundamental solutions is studied. As demonstrated, the disturbances induced by physical sources in fluids or dielectrics maintain their primary character while propagating throughout these media.

In [4], novel representations of Lagrangian acoustic field theory are developed based on an analogy between acoustics and electromagnetism. In contrast to classical spinless approaches in acoustics employing a single scalar velocity potential, the proposed representations employ vector potentials as in electromagnetism. Then, in order to take into account both the scalar and vector quantities of the acoustic field theory, a joint spinor-potential representation, which includes both the scalar and vector potentials, is developed. The proposed theory reveals and leverages a profound set of symmetries hidden in the structure of the acoustic field theory, allowing for the foundational understanding of acoustics. Based on these results, the important role played by dynamical potentials in both acoustics and electromagnetism is explored in [5]. It occurs that expressing both theories in the geometric language of a Clifford bundle over spacetime significantly clarifies the structure of each theory while illuminating their many similarities as well as their key geometric differences.

Based on the references presented above, it can easily be seen that the problem of similarities and differences between acoustics and electromagnetism still remains an active research topic among the scientific community. Therefore, we have decided to take part in this discussion and point out that the similarities between both theories result from a common source, which is the wave equation. In contrast to the usual methodology, which relies on demonstrating that the mathematical equations of acoustics and electromagnetism result in the wave equation, we have decided to show that it is possible to derive the equations of acoustics and electromagnetism from the wave equation, after setting the dimension of the solution space and assuming the existence of field helicity. Then, the control engineering processes (i.e., acoustics and electromagnetism) resulting from the wave equation can be thoroughly described and investigated, which we have done in this work. This is particularly important considering that such a unified approach can be used in the analysis of scattering from rough surfaces [6], [7]. Hopefully, the obtained results will be useful for engineers and scientists interested in both the acoustic and the electromagnetic field theory.

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II. Preliminaries

In this section, we introduce the basic notation and terminology, which is used throughout the paper. We define the Fourier transformation for the absolutely integrable function of space $f\colon \mathbb{R}^3 \to \mathbb{C}$ (i.e., $f(\mathbf{r})$ where $\mathbf{r} = [\,x \quad y \quad z\,]$)

$$\mathcal{F}(f)(\mathbf{k}) = \int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r})\, d^3r \tag{1}$$

where $\mathbf{k} = [\,k_x \quad k_y \quad k_z\,]$ denotes the wavevector and $i = \sqrt{-1}$. The same symbol, but with the wavevector argument $\mathbf{k}$, is employed to denote the Fourier transform. In our derivations, we consider acoustic and electromagnetic fields which vary in time. That is, we do not consider static fields. Then, we assume that the considered solutions to wave-propagation problems are smooth functions for which the order of partial derivatives can be changed, i.e., the second-order partial derivatives exist for the considered functions which are continuous.

Now, let us introduce the Pauli matrices [8]

$$\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \boldsymbol{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and denote zero and identity matrices as $\mathbf{0}$ and $\mathbf{I}$, respectively. The Pauli matrices satisfy the property

$$\boldsymbol{\sigma}_x^2 = \boldsymbol{\sigma}_y^2 = \boldsymbol{\sigma}_z^2 = -i\boldsymbol{\sigma}_x\boldsymbol{\sigma}_y\boldsymbol{\sigma}_z = \mathbf{I}$$

thus providing the algebra isomorphic to quaternions [9].

The wave equation for the scalar wavefunction ϕ describing propagation with the velocity c is given by

$$\Box\boldsymbol{\phi} = 0 \tag{2}$$

where $\Box = \nabla^2 - c^{-2}\partial_{tt}^2 = \nabla^2 - \partial_{\tau\tau}^2$ denotes the d'Alembert operator (i.e., the wave operator) and $\tau = ct$. Let us consider the generalization of (2) called the Klein-Gordon equation

$$(\Box + \mu^2)\boldsymbol{\phi} = 0 \tag{3}$$

where $\mu \in \mathbb{R}$ is a constant parameter. The Klein-Gordon equation [10] stems from the quantisation of the energy-momentum relation in the special relativity theory. Nowadays, (3) is regarded as a relativistic field equation for spin-0 particles. For a free spin-1/2 particles described by the Dirac equation, any component of any solution to the free Dirac equation also satisfies the free Klein–Gordon equation. Moreover, every component of every quantum field also satisfies the free Klein–Gordon equation in the quantum field theory, which means that this equation is a generic expression of the field theory. Therefore, one can consider this partial-differential equation (i.e., operator) as applicable to all free matter as well as to massless fields described by the wave equation (2) (i.e., when $\mu = 0$ in (3)). For instance, classical acoustic and electromagnetic processes can be represented by the wave equation (2) which can be considered the massless Klein-Gordon equation (3). Therefore, as it is demonstrated later, we can derive the acoustic and electromagnetic field theories from the description of the massless field based on the wave equation (2). That is, acoustics and electromagnetism, described by the Euler [11] and Maxwell's equations [12], respectively, can be derived from the wave equation (2), taking into account the dimension of the solution vector and the existence of helicity. For this purpose, we employ factorisation of the wave equation based on the properties of the Pauli matrices. Hence, we are able to demonstrate that similar properties of acoustic and electromagnetic fields result from the common roots of both theories. In the next step, we are able to derive a common representation of the system with a feedback loop for acoustics and electromagnetism, known from control engineering, which is useful for analysing the properties of both fields.

III. Field Theory

In this section, we review the general theory of linear acoustic and electromagnetic fields in order to highlight their similarities and build a unified approach. We recall the well-known conclusion that both theories imply satisfaction of the wave equation with the respective fields. The obtained formulations enable contrasting derivations of both theories, given in the next section, from the wave equation and the general field theory.

A. Acoustic Field

The acoustic field theory, derived from fluid dynamics, provides methods and tools for modelling and simulating sound waves. In the absence of external forces and sources, the Euler equations are sufficient to describe sound waves in our considerations

$$\rho\partial_t\mathbf{v} = -\nabla p \tag{4}$$

$$\kappa\partial_t p = -\nabla\cdot\mathbf{v} \tag{5}$$

where $\mathbf{v}$ and p are the acoustic velocity and pressure perturbations, respectively, whereas ρ is the equilibrium mass density, and κ is the compressibility. Equation (4) represents dynamical properties of the acoustic system, while (5) stems from the equation of continuity derived from the principle of mass conservation. In addition to (4)–(5), the constraint

$$\nabla\times\mathbf{v} = 0 \tag{6}$$

which expresses the longitudinal (i.e., curl-free) character of acoustic waves is assumed. Using the time derivative of (6), one obtains the condition

$$\partial_t(\nabla\times\mathbf{v}) = 0 \tag{7}$$

which can also be achieved by taking the curl of (4).

We can rewrite the system of equations (4)–(6) as

$$\partial_t\tilde{\mathbf{v}} = -c\nabla\tilde{p} \tag{8}$$

$$\partial_t\tilde{p} = -c\nabla\cdot\tilde{\mathbf{v}} \tag{9}$$

$$\nabla\times\tilde{\mathbf{v}} = 0 \tag{10}$$

where $\tilde{p} = \sqrt{\kappa}p$, $\tilde{\mathbf{v}} = \sqrt{\rho}\mathbf{v}$, and $c = 1/\sqrt{\kappa\rho}$ denotes the speed of sound in this case [4]. The Euler equations (8)–(9) can be written in Cartesian coordinates in the matrix form

$$\partial_\tau\boldsymbol{\alpha}\mathbf{V} + (\nabla\cdot\boldsymbol{\beta})\mathbf{V} = 0 \tag{11}$$

where

$$\mathbf{V} = \begin{bmatrix} \tilde{p} \\ \tilde{\mathbf{v}} \end{bmatrix} \tag{12}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} -\boldsymbol{\sigma}_z & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and $\boldsymbol{\beta} = [\boldsymbol{\beta}_x, \boldsymbol{\beta}_y, \boldsymbol{\beta}_z]$ denotes

$$\boldsymbol{\beta}_x = \begin{bmatrix} -i\boldsymbol{\sigma}_y & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{\beta}_y = \begin{bmatrix} \mathbf{0} & -\frac{1}{2}(\boldsymbol{\sigma}_z + \mathbf{I}) \\ \frac{1}{2}(\boldsymbol{\sigma}_z + \mathbf{I}) & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{\beta}_z = \begin{bmatrix} \mathbf{0} & -\frac{1}{2}(\boldsymbol{\sigma}_x + i\boldsymbol{\sigma}_y) \\ \frac{1}{2}(\boldsymbol{\sigma}_x - i\boldsymbol{\sigma}_y) & \mathbf{0} \end{bmatrix}.$$

Taking the divergence of (8) and the gradient of (9), one can prove that the solution to (11) must also satisfy the wave equation

$$(\mathbf{I}\Box)\mathbf{V} = 0. \tag{13}$$

The property

$$\nabla \times \nabla \times = \nabla(\nabla \cdot) - \nabla^2 \tag{14}$$

is useful in these derivations.

Let us write (11) directly in the matrix form

$$\begin{bmatrix} -\partial_\tau & -\partial_x & -\partial_y & -\partial_z \\ \partial_x & \partial_\tau & 0 & 0 \\ \partial_y & 0 & \partial_\tau & 0 \\ \partial_z & 0 & 0 & \partial_\tau \end{bmatrix} \mathbf{V} = 0. \tag{15}$$

As one can note, the first row of (15) is equivalent to (9), whereas other rows are equivalent to (8). Due to (7), one can write (15) in the form

$$\mathcal{E}\mathbf{V} = 0 \tag{16}$$

where

$$\mathcal{E} = \begin{bmatrix} -\partial_\tau & -\partial_x & -\partial_y & -\partial_z \\ \partial_x & \partial_\tau & -2\partial^2_{\tau z} & 2\partial^2_{\tau y} \\ \partial_y & 2\partial^2_{\tau z} & \partial_\tau & -2\partial^2_{\tau x} \\ \partial_z & -2\partial^2_{\tau y} & 2\partial^2_{\tau x} & \partial_\tau \end{bmatrix}. \tag{17}$$

B. *Electromagnetic Field*

Let us consider Maxwell's equations in free space without sources

$$\nabla \cdot \mathbf{D} = 0 \tag{18}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{19}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{20}$$

$$\nabla \times \mathbf{H} = \partial_t \mathbf{D} \tag{21}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric- and magnetic-field strength, respectively, whilst $\mathbf{D}$ and $\mathbf{B}$ denote the electric- and magnetic-flux density, respectively. For our considerations, we assume the constitutive relations

$$\mathbf{D} = \epsilon \mathbf{E} \tag{22}$$

$$\mathbf{B} = \mu \mathbf{H} \tag{23}$$

where ϵ and μ denote the permittivity and permeability of the electromagnetic medium, respectively.

We can rewrite the system of equations (18)–(21) as follows:

$$c\nabla \times \mathbf{F} = i\partial_t \mathbf{F} \tag{24}$$

$$\nabla \cdot \mathbf{F} = 0. \tag{25}$$

In (24)–(25), $c = 1/\sqrt{\mu\epsilon}$ denotes the speed of light in the considered medium, and $\mathbf{F}$ denotes the Riemann–Silberstein vector [13]

$$\mathbf{F} = \frac{1}{\sqrt{2}}\left(\frac{\mathbf{D}}{\sqrt{\epsilon}} + i\frac{\mathbf{B}}{\sqrt{\mu}}\right). \tag{26}$$

Based on (1), one can write (25) as

$$\mathbf{k} \cdot \mathbf{F} = 0. \tag{27}$$

This means that for the time-varying electromagnetic field, the vector $\mathbf{F}$ is orthogonal to the direction of propagation represented by the wavevector $\mathbf{k}$. Hence Gauss law (25) is a constraint to the solution of (24) which represents the dynamical properties of electromagnetic system.

Taking again into account property (14), one can prove that the solution to (24)–(25) satisfies the wave equation

$$(\mathbf{I}\Box)\mathbf{F} = 0. \tag{28}$$

The first half of Maxwell's equations (24) can be written in Cartesian coordinates as

$$\partial_\tau \mathbf{F} + (\nabla \cdot \boldsymbol{\Sigma})\mathbf{F} = 0 \tag{29}$$

where $\boldsymbol{\Sigma} = [\boldsymbol{\Sigma}_x, \boldsymbol{\Sigma}_y, \boldsymbol{\Sigma}_z]$ denotes the 3×3 spin-1 counterparts of the Pauli matrices

$$\boldsymbol{\Sigma}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \boldsymbol{\Sigma}_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \boldsymbol{\Sigma}_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If the vector $\mathbf{F}_+$ satisfies (29), i.e.,

$$\partial_\tau \mathbf{F}_+ + (\nabla \cdot \boldsymbol{\Sigma})\mathbf{F}_+ = 0 \tag{30}$$

then its complex conjugate vector $\mathbf{F}_+^*$, denoted afterwards as $\mathbf{F}_-$, satisfies

$$\partial_\tau \mathbf{F}_- - (\nabla \cdot \boldsymbol{\Sigma})\mathbf{F}_- = 0. \tag{31}$$

The solutions $\mathbf{F}_+$ and $\mathbf{F}_-$ correspond to the left- and right-handed circularly polarized waves (i.e., waves of opposite helicities) which satisfy (30) and (31), respectively.

Then, let us construct 4×4 matrices which include, as submatrices, the elements of the vector $\boldsymbol{\Sigma}$ in the bottom-right corner and which can be represented by block matrices consisting of the Pauli matrices and zeros. Additionally, we require that the obtained matrices should be asymmetrical. Then, one obtains

$$\boldsymbol{\gamma}_x = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

$$\boldsymbol{\gamma}_y = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

$$\gamma_z = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}.$$

These matrices can be written as

$$\gamma_x = \begin{bmatrix} \boldsymbol{\sigma}_y & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_y \end{bmatrix} \gamma_y = \begin{bmatrix} \mathbf{0} & -i\boldsymbol{\sigma}_z \\ i\boldsymbol{\sigma}_z & \mathbf{0} \end{bmatrix} \gamma_z = \begin{bmatrix} \mathbf{0} & -i\boldsymbol{\sigma}_x \\ i\boldsymbol{\sigma}_x & \mathbf{0} \end{bmatrix}.$$

Let us compose the vector $\boldsymbol{\gamma} = [\gamma_x, \gamma_y, \gamma_z]$ and write the equation

$$\partial_\tau \boldsymbol{\Phi} + (\nabla \cdot \boldsymbol{\gamma})\boldsymbol{\Phi} = 0 \tag{32}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} 0 \\ \mathbf{F} \end{bmatrix}. \tag{33}$$

Equations (32)–(33) are equivalent to Maxwell's equations without sources (24)–(25). Let us write (32) directly in the matrix form

$$\begin{bmatrix} \partial_\tau & -i\partial_x & -i\partial_y & -i\partial_z \\ i\partial_x & \partial_\tau & -i\partial_z & i\partial_y \\ i\partial_y & i\partial_z & \partial_\tau & -i\partial_x \\ i\partial_z & -i\partial_y & i\partial_x & \partial_\tau \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{F} \end{bmatrix} = 0. \tag{34}$$

One can note that the equation in the first row of (34) is the implementation of Gauss law (25), whereas the remaining equations describe the system dynamics (24). Thus we are able to combine the dynamics of the electromagnetic system with the constraint in a single matrix equation (34).

Using (32), (30)–(31) can respectively be written as

$$\mathcal{M}\boldsymbol{\Phi}_+ = 0 \tag{35}$$

$$\mathcal{M}^*\boldsymbol{\Phi}_- = 0 \tag{36}$$

where $\mathcal{M} = \mathbf{I}\partial_\tau + \nabla \cdot \boldsymbol{\gamma}$ is the matrix representation of Maxwell's equations, and

$$\boldsymbol{\Phi}_+ = \begin{bmatrix} 0 \\ \mathbf{F}_+ \end{bmatrix}$$

$$\boldsymbol{\Phi}_- = \begin{bmatrix} 0 \\ \mathbf{F}_- \end{bmatrix}$$

correspond to the left- and right-handed circularly polarized waves. Such a representation stems from the property $\boldsymbol{\gamma}^* = -\boldsymbol{\gamma}$.

IV. Factorisation of Wave Equation

In this section, we demonstrate that the theories of acoustic and electromagnetic fields can be derived from the wave equation with constraints, correctly assuming the dimensions of the solution space as well as taking helicity into account.

A. Acoustic Field

Let us construct the wave equation for the real 4-dimensional vector $\boldsymbol{V}$

$$(\mathbf{I}\Box)\boldsymbol{V} = 0. \tag{37}$$

One can decompose, i.e., factorize, it as follows:

$$(\mathbf{I}\Box)\boldsymbol{V} = \mathcal{E}\mathcal{E}\boldsymbol{V} = \mathcal{E}^2\boldsymbol{V} = 0. \tag{38}$$

This means that the Euler equations stem from the wave equation for the 4-dimensional acoustic vector field (without field helicity) with real components.

B. Electromagnetic Field

Let us construct the wave equation for the complex 4-dimensional vector $\boldsymbol{\Phi}$

$$(\mathbf{I}\Box)\boldsymbol{\Phi} = 0. \tag{39}$$

One can decompose, i.e., factorize, it as follows:

$$(\mathbf{I}\Box)\boldsymbol{\Phi} = \mathcal{M}\mathcal{M}^*\boldsymbol{\Phi} = \mathcal{M}^*\mathcal{M}\boldsymbol{\Phi} = 0. \tag{40}$$

This means that Maxwell's equations stem from the wave equation for the 4-dimensional vector field with complex components assuming the existence of field helicity. The vectors $\boldsymbol{\Phi}_+$ and $\boldsymbol{\Phi}_-$, corresponding to the left- and right-handed circularly polarized waves, respectively, are the solutions to the wave equation (39) and then to Maxwell's equations (35)–(36).

There are various similar results, especially those aimed at writing Maxwell's equations in the form similar to the Dirac equation. However, they differ from the presented factorization (40). Moses [14] obtained the factorization of the wave equation but without the property indicated in (40). In [15], a factorisation analogous to (40) is presented. However, the set of basis matrices $\boldsymbol{\gamma}$ includes the identity matrix, hence it does not actually resemble the Dirac equation.

V. Systems with Feedback Loop

In this section, we demonstrate that acoustics and electromagnetism can be considered continuous-time systems with feedback loops described by state–space equations of control engineering. The aim of control engineering is to control processes (i.e., dynamical systems described by partial-differential equations) in order to drive them to a desired state, often in the optimal way. This always requires understanding the nature of partial differential equations that govern the process. As demonstrated in the previous sections, acoustics and electromagnetism stem from the same hyperbolic partial differential equation, i.e., the wave equation (2). Now, we can proceed further in order to develop a unified approach representing specific cases of acoustics and electromagnetism. This allows for finding analytical solutions with using control engineering methods and for understanding causality, control and stability of acoustic and electromagnetic systems (i.e., fields).

In our approach, we are concerned with a natural response of either the acoustic or the electromagnetic system which results solely from the initial conditions with no other inputs.

A. Acoustic Field

One can write (4)–(5) in the compact form

$$\frac{\partial}{\partial t}\begin{bmatrix} p \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 & -\kappa^{-1}[\nabla \cdot] \\ -\rho^{-1}[\nabla] & 0 \end{bmatrix} \begin{bmatrix} p \\ \mathbf{v} \end{bmatrix}. \tag{41}$$

The above form of equations resembles state-space equations in control engineering. Usually, the acoustic field is considered in a discrete set of spatial points. Therefore, let us discretize (41) spatially with the use of Yee's grid for acoustics [16], [17], see Fig. 1a. We assume that the pressure locations are at the nodes in the three-dimensional (3-D) space and the velocities are located between the nodes. Hence the 3-D cuboid

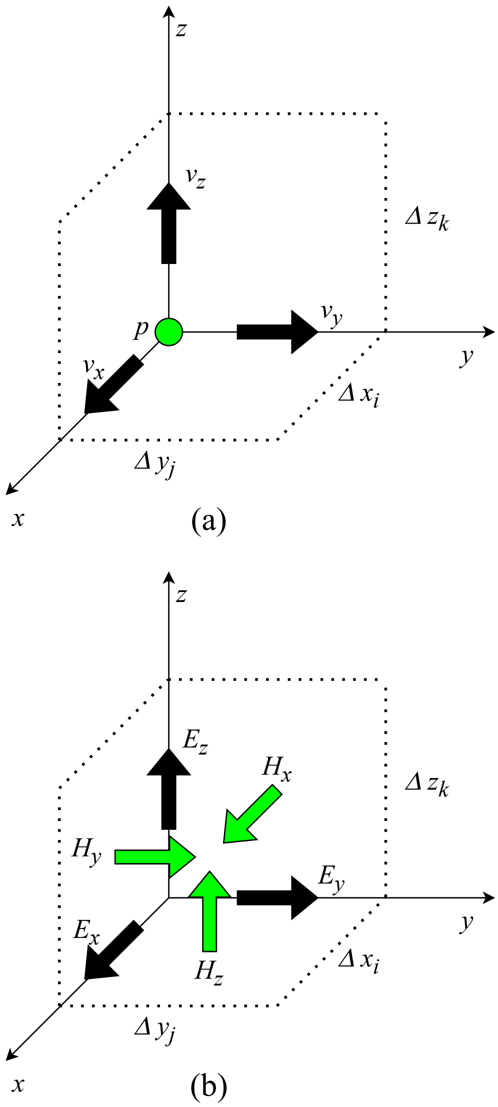

Fig. 1. Yee's grid for (a) acoustic and (b) electromagnetic fields.

V in the Cartesian space is decomposed into smaller cuboid cells, i.e.,

$$V = \{V_{i,j,k} \in \mathbb{R}^3 : [x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]\}. \quad (42)$$

In (42), $i = 0, ..., N_x - 1$, $j = 0, ..., N_y - 1$, $k = 0, ..., N_z - 1$, hence the total number of cells in the considered domain is equal to $N_x N_y N_z$, and the volume of cell $V_{i,j,k}$ is equal to $\Delta x_i \Delta y_j \Delta z_k$ (where $\Delta x_i = x_{i+1} - x_i$, $\Delta y_j = y_{j+1} - y_j$, $\Delta z_k = z_{k+1} - z_k$). Because some field components may take fixed values due to the boundary conditions, we finally assume that the components p, v_x, v_y, v_z are sampled in N_p, N_{vx}, N_{vy}, N_{vz} points, respectively. In this way the state-space matrix equation is obtained for acoustics

$$\dot{\mathbf{x}}_{ac} = \mathbf{A}_{ac}\mathbf{x}_{ac} \quad (43)$$

where

$$\mathbf{x}_{ac} = [(p)_1, ..., (p)_{N_p}, (v_x)_1, ..., (v_x)_{N_{vx}}, (v_y)_1, ..., (v_y)_{N_{vy}}, (v_z)_1, ..., (v_z)_{N_{vz}}]^T$$
$$\mathbf{A}_{ac} = \begin{bmatrix} 0 & -\kappa^{-1}[\nabla\cdot]_D^v \\ -\rho^{-1}[\nabla]_D^p & 0 \end{bmatrix}$$
$$N_v = N_{vx} + N_{vy} + N_{vz}.$$

and $[\nabla]_D^p$, $[\nabla\cdot]_D^v$ are discrete equivalents of the gradient and divergence operators for the pressure and velocity fields, respectively. In other words, the matrix operators $[\nabla]_D^p$ and $[\nabla\cdot]_D^v$ of the sizes $N_v \times N_p$ and $N_p \times N_v$, respectively, approximate the gradient and divergence operators with the difference quotients of the field values in the considered cell $V_{i,j,k}$ and the neighbouring cells [16], [17]. Obviously, the spatial discretization has to be sufficiently small in order to approximate (41) based on (43) well, and to assure the stability of the calculations.

B. Electromagnetic Field

One can write (18)–(21) in the compact form

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} 0 & \epsilon^{-1}[\nabla\times] \\ -\mu^{-1}[\nabla\times] & 0 \end{bmatrix}\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}. \quad (44)$$

Again, the above form of equations resembles state-space equations in control engineering. Because the electromagnetic field is considered in a discrete set of spatial points, as previously, we discretize (44) spatially with the use of Yee's grid for electromagnetism [16], [17], see Fig. 1b. Hence the 3-D cuboid V in the Cartesian space is decomposed into smaller cuboid cells, based on (42). Because some field components may take fixed values due to the boundary conditions, we finally assume that the components E_x, E_y, E_z, H_x, H_y, H_z are sampled in N_{ex}, N_{ey}, N_{ez}, N_{hx}, N_{hy}, N_{hz} points, respectively. Hence the following state-space matrix equation is obtained for electromagnetism

$$\dot{\mathbf{x}}_{em} = \mathbf{A}_{em}\mathbf{x}_{em} \quad (45)$$

where

$$\mathbf{x}_{em} = [(E_x)_1, ..., (E_x)_{N_{ex}}, (E_y)_1, ..., (E_y)_{N_{ey}}, (E_z)_1, ..., (E_z)_{N_{ez}}, (H_x)_1, ..., (H_x)_{N_{hx}}, (H_y)_1, ..., (H_y)_{N_{hy}}, (H_z)_1, ..., (H_z)_{N_{hz}}]^T$$
$$\mathbf{A}_{em} = \begin{bmatrix} 0 & \epsilon^{-1}[\nabla\times]_D^H \\ -\mu^{-1}[\nabla\times]_D^E & 0 \end{bmatrix}$$
$$N_e = N_{ex} + N_{ey} + N_{ez}$$
$$N_h = N_{hx} + N_{hy} + N_{hz}$$

and $[\nabla\times]_D^E$, $[\nabla\times]_D^H$ are discrete equivalents of the curl operator for the electric and magnetic fields, respectively. In other words, the matrix operators $[\nabla\times]_D^E$ and $[\nabla\times]_D^H$ of the sizes $N_h \times N_e$ and $N_e \times N_h$, respectively, approximate the curl operator with the difference quotients of the field values in

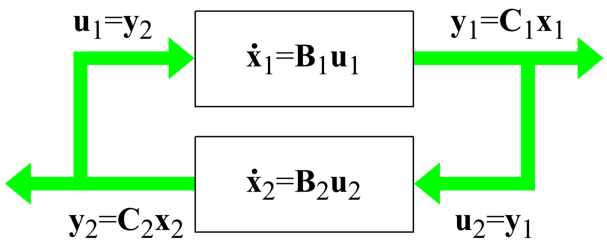

Fig. 2. System with feedback loop representing acoustics and electromagnetism.

the considered cell $V_{i,j,k}$ and the neighbouring cells [16], [17]. Again, the spatial discretization has to be sufficiently small in order to approximate (44) based on (45) well, and to assure the stability of the calculations.

C. Unified Approach to Acoustic and Electromagnetic Fields

In terms of control engineering, (43) and (45) are state-space differential equations which can be decomposed further into the system with a feedback loop, see Fig. 2.

In acoustics, the matrix and vector symbols on the scheme of the system with a feedback loop denote

$$\mathbf{x}_1 = p$$

$$\mathbf{x}_2 = \mathbf{v}$$

$$\mathbf{B}_1 = -\kappa^{-1}\mathbf{I}$$

$$\mathbf{B}_2 = -\rho^{-1}\mathbf{I}$$

$$\mathbf{C}_1 = [\nabla]_D^p$$

$$\mathbf{C}_2 = [\nabla\cdot]_D^v.$$

In electromagnetism, the matrix and vector symbols on the scheme of the system with a feedback loop denote

$$\mathbf{x}_1 = \mathbf{E}$$

$$\mathbf{x}_2 = \mathbf{H}$$

$$\mathbf{B}_1 = -\epsilon^{-1}\mathbf{I}$$

$$\mathbf{B}_2 = -\mu^{-1}\mathbf{I}$$

$$\mathbf{C}_1 = [\nabla\times]_D^E$$

$$\mathbf{C}_2 = [\nabla\times]_D^H.$$

The proposed representation of acoustic and electromagnetic systems can be further discretized in the time domain for the purpose of numerical simulations. This approach is positively verified in the case of electromagnetic simulations and presented in [18].

VI. Properties

As demonstrated, both acoustics and electromagnetism stem from the wave equation (2) which describes the causal process in terms of control engineering being the wave propagating with a constant speed. Hence the acoustic and electromagnetic fields share the same properties and solutions:

- The wave equation ensures the conservation of energy.
- The wave equation ensures the causality of solutions.
- The wavefront propagates with a constant speed.
- Analytical solutions to the state-space equations (43) and (45) can be unified and formulated as

$$\mathbf{x} = \mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 \tag{46}$$

  where $\mathbf{A}$, $\mathbf{x}$ are either $\mathbf{A}_{ac}$, $\mathbf{x}_{ac}$ or $\mathbf{A}_{em}$, $\mathbf{x}_{em}$ for acoustics or electromagnetism, respectively, $\mathbf{x}_0 = \mathbf{x}(t_0)$ is the initial condition for the state vector, and $e^{\mathbf{A}t}$ is the matrix exponential [19], [20].
- The solutions to the wave-propagation problems in acoustics and electromagnetism are unique due to (46).

VII. Conclusion

Similarities and differences between acoustic and electromagnetic field theories remain an open research problem. Therefore, we develop a unified approach to acoustics and electromagnetism which employs control engineering methods for their analysis and modelling. Both theories are derived from the wave equation representing massless fields, and then represented as a system with a feedback loop in control engineering. This approach enables further investigations of the properties for acoustics and electromagnetism simultaneously. Our results indicate several intriguing directions for future research that may shed light on the issues related to each theory. For instance, the existence of spin in acoustics has recently been discovered and our unified approach does not consider this phenomena. The conservation of momentum is not discussed because, in electromagnetism, there exist two momentum definitions of Minkowski and Abraham which reduce to the same definition only for a vacuum. Furthermore, depending upon the performed experiment, one of the two definitions of the electromagnetic wave momentum agrees with the experimental results.

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