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# Numerical Implementation of the Fictitious Domain Method for Elliptic Equations

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***Abstract*—In this paper, we consider an elliptic equation with strongly varying coefficients. Interest in the study of these equations is connected with the fact that this type of equation is obtained when using the fictitious domain method. In this paper, we propose a special method for the numerical solution of elliptic equations with strongly varying coefficients. A theorem is proved for the rate of convergence of the iterative process developed. A computational algorithm and numerical calculations are developed to illustrate the effectiveness of the proposed method.**

***Keywords*—elliptic equation, Dirichlet problem, equation with rapidly varying coefficients, computational algorithm, iterative process, fictitious domains method, boundary conditions**

## I. Introduction

THE fictitious domain method is efficient for the numerical solution of elliptic equations in irregular shape domains. In paper [1] an efficient (with respect to the number of operations) alternately-triangular scheme of second order accuracy for the numerical solution of an elliptic equation is proposed. In [2], a modified alternate-triangular iterative method with Chebyshev parameters for the solution of the Dirichlet problem for elliptic equations of second order accuracy is built. In V.I. Lebedev's monograph [3], the application of the method of composition for finding solutions for eigenvalue problems, time-dependent problems, the Dirichlet problem for the biharmonic equation, and grid problems is considered. In [4], the difference stationary problem for the Poisson equation with piecewise constant coefficients in subdomains is considered. Poisson equation at the interface can be approximated in a special way, i.e. difference equation coefficients are chosen as a quotient, in the denominator of which is the sum of the coefficients in subdomains. A two-step iterative process based on the method of dividing the area is built.

Papers of Bugrov A.N., Konovalov A.N., Smagulov Sh.S., Orunkhanov M.K., Kuttykozhayeva Sh.N. [5-9] are devoted to the fictitious domain method for the equations of mathematical physics. In these references, they study different modifications of the fictitious domain method with continuation upon low-order coefficients for the Poisson equation. Estimates of the method's convergence rate depending on a small parameter were obtained.

In this paper, we propose a special method for the numerical solution of elliptic equations with strongly varying coefficients. The basis of the suggested method is in the special replacement of variables which reduces the problem with second order discontinuous coefficients to the problem with first order discontinuous coefficients. An iterative process with two parameters taking into account the ratio of the coefficients of the equation in subdomains is built. A theorem for the rate of convergence of the developed iterative process is proved. A computational algorithm is developed and numerical calculations to illustrate the effectiveness of the proposed method are conducted.

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## II. Statement of the Problem

Let $\Omega$ be a bounded domain in $R^2$ with piecewise smooth boundary $\partial\Omega$. For definiteness, we consider $\Omega = Q_1 \cup Q_2, Q_1 \cap Q_2 = \Gamma,$ where $Q_2$ is the interior subdomain. In $\Omega$, consider the elliptic equation

$$-div(k\nabla u) = f(\vec{x}), \vec{x} \in \Omega \tag{1}$$

with the boundary conditions

$$u(\vec{x}) = 0, \vec{x} \in \partial\Omega, \tag{2}$$

where

$$k(\vec{x}) = \begin{cases} k_1 = const, \vec{x} \in Q_1, \\ k_2 = const, (\vec{x}), x \in Q_2. \end{cases}$$

The function $f(\vec{x})$ is assumed to belong to the Hilbert space of real functions $L_2(\Omega)$, and in subdomains, it is defined as follows:

$$f(\vec{x}) = \begin{cases} f^{(1)}(x), x \in Q_2, \\ 0, x \in Q_1. \end{cases}$$

We make the replacement of variables in (1) in the form $u = 2v/k_1$, and after simple transformations, we obtain

$$\Delta v + div(\omega\nabla v) = -f(\vec{x}), \tag{3}$$

where $\omega = \frac{2k(x)}{k_1} - 1.$ Let us designate $\theta = \frac{2k_2}{k_1} - 1.$

We introduce the notation $\vec{p} = \left( \omega\frac{\partial v}{\partial x_1}, \omega\frac{\partial v}{\partial x_2} \right)$ and write the equation (3) as the following system of equations:

$$\begin{cases} \Delta v + \nabla\vec{p} = -f(\vec{x}), \\ p_1/\omega - \frac{\partial v}{\partial x_1} = 0, \\ p_2/\omega - \frac{\partial v}{\partial x_2} = 0. \end{cases} \tag{4}$$

## III. Computational Algorithm

For the numerical solution of equations (4) with the boundary conditions $v\big|_{\partial\Omega}=0$, we consider the iterative method:

$$Bv_t^{n+1}+\Delta_h v^{n+1}+\nabla_h \vec{p}^{\,n+1}=-f(\vec{x}), \tag{5}$$

$$\beta(\vec{p}^{\,n+1}-\vec{p}^{\,n})+\frac{\vec{p}^{\,n+1}}{\omega}-\nabla_h v^{n+1}=0,$$

where $B$ is an operator of the iterative method, $\beta$ is an iterative parameter, the index $h$ is the difference analog of the differential operator. The operator $B$ in the iterative method (5) is chosen as follows:

$$B=(1-\tau)\Delta_h-\tau div_h(\rho\nabla_h), \tag{6}$$

where $\rho=(\beta+1/\omega)^{-1}$.

Assume that $v^0\in \overset{0}{W}{}_2^1(\Omega)$ and $p^0=\nabla q$, where $q\in \overset{0}{W}{}_2^1$. In particular, this condition is satisfied if $(v^0,\,p^0)=0$.

Hereinafter, we assume that $B>0$ in $\overset{0}{W}{}_2^1$. For this, the following equation must hold:

$$1-\tau-\frac{\tau}{\beta}>0. \tag{7}$$

In this case, $B$ in $\overset{0}{W}{}_2^1$ satisfies the operator inequality:

$$-\chi_1\Delta\le B\le-\chi_2\Delta, \tag{8}$$

the constants $\chi_1$ and $\chi_2$ can be chosen independent on $\theta\ge1$.

Substituting the operator $B$, defined as (6), to (5), we obtain

$$\Delta_h v^{n+1}=F(x), \tag{9}$$

$$\vec{p}^{\,n+1}=\beta\rho\vec{p}^{\,n}+\rho\nabla_h v^{n+1}, \tag{10}$$

where

$$F(\vec{x})=(1-\tau)\Delta_h v^n-\tau div_h(\beta\rho\nabla_h v^n)-\tau div_h(\beta\rho\vec{p}^{\,n}) \tag{11}$$

We present an algorithm of the numerical implementation of the method (9), (10). One step of the iterative method (9), (10) consists in finding values $v^{n+1}$ using the known values of $v^n$, $\vec{p}^{\,n}$. This requires to solve the Dirichlet problem for the Poisson equation (9) in $\Omega$.. Then, the value of $\vec{p}^{\,n+1}$ is defined by the known values of $\vec{p}^{\,n}$ and $v^{n+1}$ using the formula (10).

## IV. The Study of Convergence

Let us prove some auxiliary estimates that will be needed in the study of the iterative method. Let $H(Q_2)$ be the closure, in $W_2^1(Q_2)$, of the set of smooth functions orthogonal to the unit on $\Gamma$, and $H(Q_1)$ be the closure, in $\overset{0}{W}{}_2^1(Q_1)$, of a set of smooth functions vanishing on $\Gamma$. We introduce the norm in $H(Q_i)$ as follows:

$$\|v\|_{H(Q_i)}=\|\nabla v\|_{Q_i}=\left(\int_{Q_i}|\nabla v|^2\,dx\right)^{1/2}.$$

Let $\varphi$ be defined on $\Gamma$, and $(\varphi,1)_\Gamma=\int_\Gamma\varphi ds=0$. We denote

$$\|\varphi\|_{-1/2Q_i}=\sup_{\eta\in H(Q_i)}(\varphi,\eta)_\Gamma\Big/\|\nabla\eta\|_{Q_i} \tag{12}$$

### A. Lemma 1

Let $v_2\in W_2^1(Q_2)$ be a generalized solution of the problem

$$\Delta v_2=0,\ \vec{x}\in Q_2 \tag{13}$$

$$\frac{\partial v_2}{\partial n}\Big|_\Gamma=0$$

and $v_1$ be a generalized solution of the problem

$$\Delta v_1=0,\ \vec{x}\in Q_1 \tag{14}$$

$$\frac{\partial v_1}{\partial n}\Big|_\Gamma=0,\ v_1\big|_{\partial\Omega}=0.$$

then

$$\|\nabla v_2\|_{Q_2}^2\le C_3\|\nabla v_1\|_{Q_1}^2 \tag{15}$$

where $C_3$ does not depend on $\varphi$.

### B. Proof

We introduce the norm

$$\|\varphi\|_{-1/2\Gamma}=\sup_{\psi\in W_2^{1/2}(\Gamma)}(\varphi,\psi)_\Gamma/\|\psi\|_{1/2\Gamma} \tag{16}$$

A generalized solution of the problem (13) is a function in $H(Q_2)$ which satisfies the relation

$$(\nabla v_2,\nabla\eta)=(\varphi,\eta)_\Gamma,\ \forall\eta\in H(Q_2) \tag{17}$$

According to the embedding theorem, the right side of (17) is a bounded linear functional in $H(Q_i)$. According to the Riesz's theorem, there exists a function $v_0\in H(Q_2)$ such that

$$(\varphi,\eta)_\Gamma=(\nabla v_0,\nabla\eta)_{Q_2}$$

and

$$\|\nabla v_0\|=\sup_{\eta\in H(Q_2)}\frac{(\varphi,\eta)_\Gamma}{\|\nabla\eta\|_{Q_2}} \tag{18}$$

Then, using (17) and (18) we have

$$v_2=v_0\ \text{ and }\ \|\nabla v_2\|_{Q_2}=\|\varphi\|_{-1/2Q_2}.$$

Similarly, we prove that

$$\|\nabla v_1\|_{Q_1}=\|\varphi\|_{-1/2Q_1}.$$

Norms (12) are equivalent for $i=1,2$. In fact, according to the embedding theorems, we have the chain of inequalities:

$$\frac{|(\varphi,\psi)_\Gamma|}{\|\nabla\eta\|_{Q_i}}\le\frac{\|\varphi\|_{-1/2\Gamma}\|\psi\|_{-1/2\Gamma}}{\|\nabla\eta\|_{Q_i}}\le c\|\varphi\|_{-1/2,\Gamma}.$$

On the other hand, every function $\psi\in W_1^{1/2}(\Gamma)$ (in the case of $i=2$, the function $\psi$ satisfies the condition $(\psi,1)_\Gamma=0$) can be extended to $Q_i$ so that the extended function $\tilde{\psi}$ belongs to $H(Q_i)$, and

$$\|\nabla\tilde{\psi}\|_{Q_i} \le c\|\psi\|_{-1/2\Gamma}$$

Thus,

$$\frac{|(\varphi,\psi)_\Gamma|}{\|\psi\|_{-1/2\Gamma}} \le c\frac{|(\varphi,\psi)_\Gamma|}{\|\nabla\tilde{\psi}\|_{Q_i}}$$

It follows that the norms $\|\varphi\|_{-1/2Q_i}$ are equivalent. According to the equality $\|\nabla v_i\|_{Q_i} = \|\varphi\|_{-1/2Q_i}$, we obtain the estimate (15). The lemma is proven.

Now we estimate the rate of convergence of the method (9), (10). Let us denote

$$\{y, r\} = \{y^n, r^n\} = \{v - v^n,\ p - p^n\},$$
$$\{\hat{y}, \hat{r}\} = \{y^{n+1}, r^{n+1}\}.$$

Then, equations (5) can be rewritten as

$$(By_t, v) + (\nabla_h\hat{y}, \nabla_h v) + (\nabla_h\hat{r}, v) = 0,\ \forall v \in \overset{0}{W}{}_2^1, \qquad (19)$$

$$\beta\tau r_t + \hat{r}/\omega - \nabla_h\hat{y} = 0, \qquad (20)$$

$$\{y^0, r^0\} \in \overset{0}{W}{}_2^1 \times L_2,\ \text{where } y_t = (\hat{y} - y)/\tau.$$

A function $\psi$ in $L_2$ is a piecewise gradient function if it can be represented in the form

$$\psi = \nabla g_i \text{ in } Q_i;\ \text{where } g_i \in W_2^1(Q_i) \qquad (21)$$
$$g_i\big|_{\partial\Omega\cap\partial\Omega_i} = 0,\ i = 1, 2, ..., N,$$

and a function $\psi$ is a gradient function, if it is in the form

$$\psi = \nabla g \text{ in } \Omega,\ \text{where } g \in \overset{0}{W}{}_2^1(\Omega).$$

Since $p^0 = \nabla g,\ g \in \overset{0}{W}{}_2^1$ and $\omega$ is a piecewise constant, then $r^0$ is a piecewise gradient function.

We take the inner product of both sides of the equation (20) with $2\tau\hat{r}$ in $L_2$ and set $v = 2\tau\hat{y}$ in relation (19). Adding these equalities, we have

$$\|\hat{y}\|_B^2 - \|y\|_B^2 + \tau^2\|y_t\|_B^2 + 2\tau\|\nabla_h\hat{y}\|^2 + \beta\tau\|\hat{r}\|^2 - \beta\tau\|r\|^2 + \beta\tau^3\|r_t\|^2 + \frac{2\tau}{\omega}\|\hat{r}\|^2 = 0. \qquad (22)$$

Let us investigate the form of $r^n$. Since

$$\hat{\tau} = \frac{\beta}{\beta + 1/\omega}r + \frac{1}{\beta + 1/\omega}\nabla\hat{y}$$

and $r$ is a piecewise gradient function, then $\hat{\tau}$ is a piecewise gradient function. Thus, all $r^n$ are piecewise gradient functions.

Let $G$ be the space of piecewise gradient functions, and $G_1$ be the space of gradient functions. It is obviously that $G_1 \subseteq G$. Let us show that there is a strict embedding $G_1 \subset G$, and we will show the orthogonality, in $L_2$, of the complement $G_1$ to $G$. If $\psi$ is orthogonal, in $L_2$, to all elements of $G_1$, then for every element $\nabla q \in G_1$ we have $(\psi, \nabla q)_\Omega = 0$. If the function $\nabla g$ is sufficiently smooth and it has a support in $Q_i$ then

$$(\psi, \nabla g)_\Omega = (\psi, \nabla g)_{Q_i} = -(div\,\psi, g)_{Q_i} = -(\Delta g_i, g)_{Q_i} = 0.$$

Since $g$ is arbitrary, the last relation implies

$$\Delta g_i = 0 \text{ in } Q_i. \qquad (23)$$

It is clear that the relation holds in every $Q_i, i = 1, 2..$. Thus, the element $\psi \in G$, orthogonal to all elements of $G_1$, is represented in the form (21), where $q_i$ is a harmonic function in $Q_i$. Let us find conditions which must be satisfied by $\psi$ being orthogonal to $G_1$ on $\Gamma$. Let $\nabla q \in G_1$, then

$$0 = (\psi, \nabla q)_\Omega = (\nabla q_1, \nabla q)_{Q_1} + (\nabla q_2, \nabla q)_{Q_2} =$$
$$= \int_\Gamma g\frac{\partial q_1}{\partial n_1}ds + \int_\Gamma g\frac{\partial q_2}{\partial n_2}ds = \int_\Gamma g\left(\frac{\partial q_1}{\partial n_1} - \frac{\partial q_2}{\partial n_1}\right)ds$$

(here $n_i$ are vectors of the outward normal on $\partial Q_i$); i.e., the values of the normal components $\psi_1 = \nabla q_1$ and $\psi_2 = \nabla q_2$ on $\Gamma$ are the same. Thus, the normal component of the vector function $\psi$ is continuous (in the integral sense) when passing through $\Gamma$. This implies that orthogonal, in $L_2$, complement $G_2$ of the space $G_1$ to the space $G$ consists of all functions of the form (21), the normal component of which is continuous when passing through the adjacent border, and functions $g_i$ forming them are harmonic in $Q_i$.

Let us continue studying the convergence of the iterative method (9), (10). As it was discovered before, $\hat{r} \in G$. Let us represent $\hat{r}$ in the form $\hat{r} = \hat{q} + \hat{h}$, where $\hat{q} \in G_1$, and $\hat{h} \in G_2$. In this case, (19) takes the form

$$(By_t, v) + (\nabla\hat{y}, \nabla v) + (\hat{q}, \nabla v) + (\hat{h}, \nabla v) = 0, \qquad (24)$$

for $\forall v \in \overset{0}{W}{}_2^1$.

Last scalar product in (24) vanishes because $\nabla v \in G_1$. Dividing both sides of (24) by $\|\nabla v\|$ and assessing the member containing $\hat{q}$, we obtain

$$\frac{|(\hat{q}, \nabla v)|}{\|\nabla v\|} \le \frac{|(By_t, v)|}{\|\Delta v\|} + \frac{|(\nabla\hat{y}, \nabla v)|}{\|\nabla v\|} \le \sqrt{\chi_2}\|y_t\|_B + \|\nabla_h\hat{y}\|.$$

Since the right side of this inequality does not depend on $v \in \overset{0}{W}{}_2^1$ and $\hat{q} \in G_1$, i.e. it is represented in the form $\hat{q} = \nabla g (g \in \overset{0}{W}{}_2^1)$, then taking sup by $v$ on the left side of the inequality, we obtain

$$\|\hat{q}\| \le \sqrt{\chi_2}\|y_t\|_B + \|\nabla_h\hat{y}\|,$$

where $\|\nabla_h\hat{y}\| = \|\hat{y}\|_1$. Let us square both sides of this inequality and estimate the right side:

$$\|\hat{q}\|^2 \le 2(\chi_2\|y_t\|_B^2 + \|\nabla_h\hat{y}\|^2).$$

We multiply this inequality by $\beta\tau^2\lambda$ ($\lambda>0$ is arbitrary) and add it to (22). As a result, we have

$$\|\hat{y}\|_B^2+\tau^2(1-2\beta\lambda\chi_2)\|y_t\|_B^2+2\tau(1-\beta\tau\lambda)\|\nabla_h\hat{y}\|^2+ \\ +\beta\tau^2\lambda\|\hat{q}\|^2+2\tau\left(\hat{r},\frac{\hat{r}}{\omega}\right)+\beta\tau\|\hat{r}\|^2\le\|y\|_B^2+\beta\tau\|r\|^2 \quad (25)$$

We estimate the scalar product $(\hat{r},\hat{r}/\omega)$. For any $\delta$, $0<\delta<1$ the following inequality holds:

$$\left(\frac{\hat{r}}{\omega},\hat{r}\right)\ge\left(\frac{\hat{q}}{\omega},\hat{q}\right)+\left(\frac{\hat{h}}{\omega},\hat{h}\right)-2\left|\left(\frac{\hat{q}}{\omega},\hat{h}\right)\right|\ge \\ \ge(1-\delta)\left(\frac{\hat{h}}{\omega},\hat{h}\right)+\left(1-\frac{1}{\delta}\right)\left(\frac{\hat{q}}{\omega},\hat{q}\right) \quad (26) \\ \ge(1-\delta)\left[\|\hat{h}\|_{Q_1}^2+\frac{1}{\theta}\|\hat{h}\|_{Q_2}^2\right]+\left(1-\frac{1}{\delta}\right)\|\hat{q}\|^2$$

Since $\hat{h}\in G$, according to Lemma 1, we have the estimate:

$$\|\hat{h}\|_{Q_2}\le c_3\|\hat{h}\|_{Q_1}^2,$$

thus,

$$\|\hat{h}\|_\Omega^2=\|\hat{h}\|_{Q_1}^2+\|\hat{h}\|_{Q_2}^2\le(1+c_3)\|\hat{h}\|_{Q_1}^2,$$

therefore, (26) yields

$$\left(\frac{\hat{r}}{\omega},\hat{r}\right)\ge c_4(1-\delta)\|\hat{h}\|^2+\left(1-\frac{1}{\delta}\right)\|\hat{q}\|^2\ c_4=(1+c_3)^{-1}.$$

Using the last inequality, we reduce (25) to the form

$$\|\hat{y}\|_B^2+\tau^2(1-2\beta\lambda\chi_2)\|y_t\|_B^2+2\tau(1-\beta\tau\lambda)\|\hat{y}\|_1^2+ \\ +\beta\tau\|\hat{r}\|^2+\beta\tau^2\lambda\|\hat{q}\|^2+2\tau(1-\delta)c_4\|\hat{h}\|^2+ \quad (27) \\ +2\tau\left(1-\frac{1}{\delta}\right)\|\hat{q}\|^2\le\|y\|_B^2+\beta\tau\|r\|^2$$

We fix $\beta>0$ and choose $\tau>0$ so that for any $\theta>1$ the condition $\beta>0$ holds. We choose $\lambda$ satisfying

$$1-2\beta\lambda\chi_2>0,\ 1-\beta\tau\lambda>0,$$

and we set $\delta=\dfrac{4}{4+\beta\tau\lambda}<1$. Then

$$\beta\tau^2\lambda+2\tau(1-1/\delta)=\beta\tau^2\lambda-2\tau\frac{\beta\tau\lambda}{4}=\frac{\beta\tau^2\lambda}{2},\ 1-\delta=\frac{\beta\tau\lambda}{4+\beta\tau\lambda}.$$

The inequality (27) for such $\delta$ is in the form

$$\left(1+\frac{c_5\tau}{\chi_2}\right)\|\hat{y}\|_B^2+\beta\tau(1+c_6\tau)\|\hat{r}\|^2\le\|y\|_B^2+\beta\tau\|r\|^2, \quad (28)$$

where $c_5=\tau(1-2\beta\lambda\chi_2)$, $c_6=\min\left\{\dfrac{\lambda}{2},\ \dfrac{2c_4\lambda}{4+\beta\tau\lambda}\right\}$.

It is obvious that the constants $\beta,\tau,\chi_2,\lambda$ can be selected the same for all $\theta$, $1\le\theta\le\infty$. Thus, we have proved the following theorem.

### C. Theorem 1

For any $\beta>0$ there exists $\bar{\tau}=\bar{\tau}(\beta)$ independent on $\omega\ge1$ such that $-\chi_1\Delta\le B\le-\chi_2\Delta$ for $\tau\le\bar{\tau}$, the constants $\chi_1$, $\chi_2$ do not depend on $\omega$.

In this case, the iterative process (9), (10) converges at a geometric rate, and speed of convergence does not depend on $\omega$.

### D. Remark

It is obvious that Theorem 1 holds when $Q_2=\bigcup_1^N\Omega_i'$ or when $Q_1=\bigcup_1^N\Omega_i'$. In this case, subregions $\Omega_i'$ should be typologically separable with piecewise smooth boundaries. In the first case, the parameter $\omega$ do not necessarily match in the subdomains $\Omega_i'$ and $\Omega_j'$.

## V. Numerical Calculations

Using the method described above, the test problem (1) - (2) was solved. The subdomain $Q_2$ was chosen in the form of a square $Q_2=\{x_{1,k_1}\le x_1\le x_{1,k_2};\ x_{2,m_1}\le x_2\le x_{2,m_2}\}$, where $x_{1,k_1}=0{,}25$, $x_{1,k_2}=0{,}75$, $x_{2,m_1}=0{,}25$, $x_{2,m_2}=0{,}75$. The area $\Omega$ covers the subdomain $Q_2$, $\Omega=\{0\le x_1\le1;\ 0\le x_2\le1\}$.

The subdomain $Q_1$ is defined as $Q_1=\Omega\setminus Q_2$. The right side is defined in $Q_2$ as follows:

$$f(x_1,x_2)=2(x_2^2-(x_{2,m_1}+x_{2,m_2})x_2+x_{2,m_1}x_{2,m_2})+ \\ 2(x_1^2-(x_{1,k_1}+x_{1,k_2})x_1+x_{1,k_1}x_{1,k_2})$$

where $x_{1,k_1}=0{,}25$, $x_{1,k_2}=0{,}75$, $x_{2,m_1}=0{,}25$, $x_{2,m_2}=0{,}75$.

In the subdomain $Q_1$ the function $f(x_1,x_2)=0$. The iterative parameter $\tau$ is chosen as $\tau=10^{-3}\div10^{-5}$, the parameter $\beta$ is determined so as to satisfy the condition (7). It is necessary to follow the sign of the parameter $\omega$ in the subdomains since $-1\le\omega\le1$.

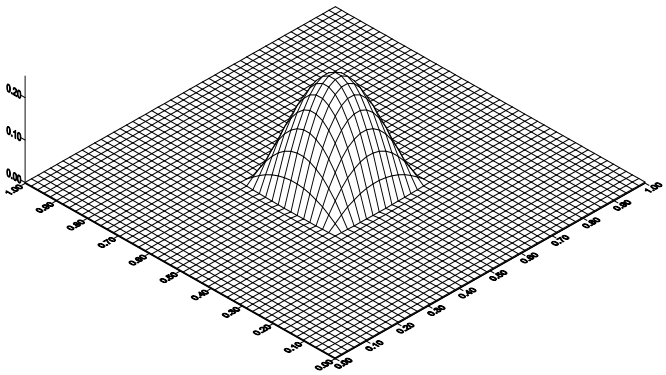

Fig. 1. Graph of the exact solution at the grid nodes 101x101

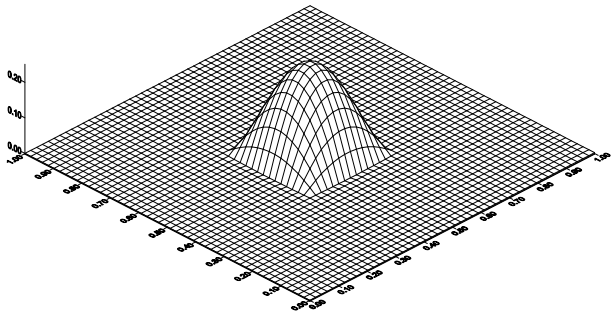

Fig. 2. Graph of the approximate solution at grid nodes 101x101

The problem for the elliptic equation with strongly varying coefficients was solved using the fictitious domain method, following the higher coefficients. Figures 1-2 show the results of the exact and the approximate solutions at grid nodes 101x101, respectively.

In the calculations, the uniform mesh sizes of 101x101, 501x501, and 1001x1001 were used. To carry out numerical experiments on a fine grid, a numerical experiment was conducted on a supercomputer URSA based on 128 quad-core processors Intel Xeon series E5335 2.00GHz at Al-Farabi Kazakh National University. The developed method is based on building a computational algorithm for the elliptic equation with strongly varying coefficients. The developed algorithm uniformly converges for a certain amount of iterations, and the results were obtained with an accuracy of $10^{-10}$ . The results of numerical experiments were visualized in the modeling package named Surfer.

## REFERENCES


[1] A. A. Samarskiy, “About One Economic Algorithm of Numerical Computation of Differential and Algebraic Equation System,” *Journal of Computation Mathematics and Mathematical Physics*, vol. 4, no. 3, pp. 580-585, 1964 (in Russian).

[2] A. B. Kucherov and E. S. Nikolayev, “Alternately-Triangular Method of Mesh Elliptical Equiations Solution in Rectangle,” *Journal of Computation Mathematics and Mathematical Physics*, vol. 16, no. 5, pp. 1164-1174, 1976.

[3] V. N. Lebedev, “Method of Composition,” Moscow: OBM. AS USSR, 1986.

[4] E. A. Volkov, ”About Methods Solving Difference Equations for Sectionally-Uniform Media and With the Right Part of Given along Curve”. *DAN USSR*, vol. 283, no. 2, pp. 274-277, 1985.

[5] A. N. Bugrov, A. N. Konovalov, and V. A. Scherbak, “Fictitious Domain Method in Plane Elastic Static Problems Numerical Methods of Mechanics Of Continua,” Novosibirsk, vol. 5, no. 1, pp.20-30, 1974.

[6] A. N. Konovalov, “Fictitious Domain Method in Torsion Problems Numerical Methods of Mechanics Of Continua,” Novosibirsk, vol. 4, no. 2, pp. 109-115, 1973.

[7] Sh. S. Smagulov, “Fictitious Domain Method for Boundary Problem of Navier-Stokes Equation,” Novosibirsk: Ed. V.Ts. SO AN USSR, no. 68, pp. 68-73, 1979.

[8] M. K. Orunkhanov and Sh. S. Smagulov, “Fictitious Domain Method for Navier-Stokes Equation in Terms of Flow Function and Vorticity with Inhomogeneous Boundary Conditions Calculation Technologies,” Novosibirsk: SO RAN, vol. 5, no.3, pp. 46-53, 2000.

[9] Sh. N. Kuttykozhaeva, “Fictitious Domain Method for Navier-Stokes Equation,” *Vestnik KazGU. Section Math., Mech., Inf.*, no 13, pp. 54-59, 1998.