

Some Contraction Methods for Locating and Finding All the DC Operating Points of Diode-Transistor Circuits

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Abstract—The paper is focused on the analysis of diode-transistor circuits having multiple DC solutions (operating points) and brings two methods enabling us to find all the solutions. The first method contracts and eliminates some hyperrectangular regions where the solutions are sought. It is based on the idea of framing of the nonlinear functions appearing in the mathematical description of the circuit by linear ones and exploits the Woodbury formula in matrix theory. The other method finds quickly and easily preliminary bounds on the location of all the solutions. The method employs some monotonic functions and generates convergent sequences leading to a shrunk hyperrectangle that contains all the solutions. Both the proposed methods are rigorously proved. They constitute the core of an algorithm which efficiently finds all the DC operating points of diode-transistor circuits. It is illustrated via numerical examples.

Keywords—Computational methods, DC solutions, diode-transistor circuits, multiple solutions.

I. INTRODUCTION

CIRCUITS having multiple DC solutions (operating points) are common and often encountered in electronic devices. Therefore computing all the DC solutions is a basic question of the analysis and design of nonlinear circuits. Even though there is a wide variety of nonlinear circuits, the transistor ones occupy a special place in electronics. Finding all the DC solutions is a difficult task in circuit simulations. Numerous methods have been proposed for tackling this problem over the last few decades [1]–[22]. Unfortunately, the methods are very time-consuming and enable us to efficiently analyse only rather small size circuits. Among the various methods concerning this problem, the most commonly used are based on piecewise-linear approximations and computation techniques e.g. [1]–[3], [5], [7]–[14].

The circuits described by the equations containing original smooth (not piecewise-linear) nonlinearities are more difficult for the analysis. Only a few methods have been proposed in this area, e.g. [15]–[23]. The paper [15] offers a simulator to find the multiple DC solutions, based on the Newton homotopy method, employing the transient analysis of SPICE. In [16] a new approach to interval method is developed and applied to finding the set of all real solutions of nonlinear simultaneous

equations. A method based on transfer characteristics and driving point characteristics for the analysis of circuits having multiple DC operating points is proposed in [17]. The work [18] offers an algorithm that employs an interval analysis, the dual simplex method and the contraction method. Threading homotopy is developed in [19] for finding multiple operating points automatically. An approach based on the partitioning of the original circuit in subcircuits and the construction of an oriented dependency graph is proposed in [21].

Not all of the described above methods guarantee finding all the DC solutions, although as a rule they find them.

The idea of successive contraction, division and elimination of some hyperrectangular regions, where the solutions are sought [4]–[6], [8], [20], [22], [23], is very useful in this field and enables us to find all the DC solutions both in piecewise-linear and smooth (not piecewise-linear) circuits. This paper proposes a method enabling us to contract the hyperrectangular regions. The method employs the idea of framing the nonlinear function by linear ones [8] and utilizes the Woodbury formula in matrix theory [24]. Furthermore, a method is proposed for finding preliminary bounds on the location of all the solutions quickly and easily. Both the methods form the core of an algorithm which guarantees finding all the DC solutions. This algorithm is efficient and enables us to analyse larger circuits without piecewise-linear approximation.

Let us consider a circuit consisting of linear resistors, diodes, bipolar transistors, and independent voltage sources. The transistors are characterized by the Ebers-Moll model with small contact resistors R_E , R_C , R_B . The circuit can be described by the Sandberg-Willson equation [25]

$$\hat{f}(v) + Av - b = 0 \quad (1)$$

where $v = [v_1 \dots v_n]^T$, $b = [b_1 \dots b_n]^T$, $A = [a_{ij}]_{n \times n}$,

$\hat{f}(v) = [\hat{f}_1(v_1) \dots \hat{f}_n(v_n)]^T$, where $\hat{f}_i(v_i)$, $i = 1, \dots, n$, are the currents flowing through the diodes included in the Ebers-Moll model and individual ones. The function $\hat{f}_i(v_i)$ that is usually specified by $K_i(e^{\lambda v_i} - 1)$ will be approximated as follows

$$\hat{f}_i(v_i) = \begin{cases} \Delta_i v_i & \text{for } v_i < w \\ K_i(e^{\lambda v_i} - 1) & \text{for } v_i \geq w \end{cases} \quad (2)$$

where $w = 0.25$ and $\Delta_i = \frac{1}{w} K_i(e^{\lambda w} - 1)$. Thus, for $v_i < w$ the graph of the function is a straight line with very small slope Δ_i . It can be proved that under very slight

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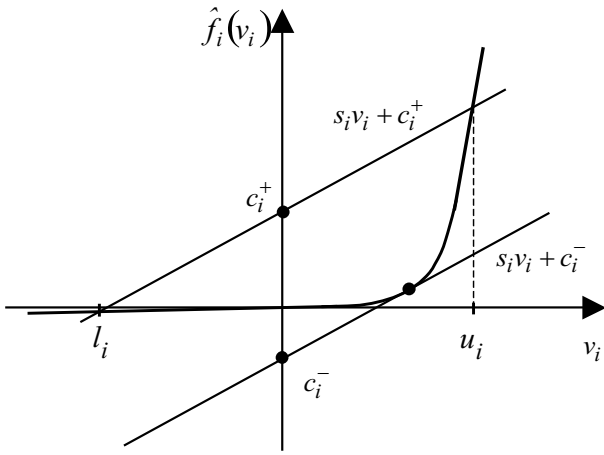


Fig. 1. Framing of the nonlinear function $\hat{f}_i(v_i)$.

restrictions the diagonal elements of the matrix \mathbf{A} are positive ($a_{ii} > 0, i = 1, \dots, n$).

We wish to find all the solutions to the equation (1) which satisfy the constraints: $-E \leq v_i \leq E, i_i \leq I_i, i = 1, \dots, n$, where E is the sum of all voltage sources acting in the circuit, whereas I_i is the forward burnout current of i -th diode. Hence, taking into account the equation (2), we obtain $\tilde{l}_i \leq v_i \leq \tilde{u}_i$, where $\tilde{l}_i = -E, \tilde{u}_i = \min \left\{ E, \frac{1}{\lambda} \ln \left(\frac{I_i}{K_i} = 1 \right) \right\}$, $i = 1, \dots, n$. Thus, we seek all the solutions to equation (1) in the n -dimensional rectangular (hyperrectangular) region $[\tilde{\mathbf{l}}, \tilde{\mathbf{u}}] = [\tilde{l}_1, \tilde{u}_1] \times \dots \times [\tilde{l}_n, \tilde{u}_n]$, $\tilde{l}_i \leq v_i \leq \tilde{u}_i, i = 1, \dots, n$. To find all the solutions, we apply the idea of successive contraction, division, and elimination as follows. We shrink the region using a contraction method so that the new one contains the same solutions. If the shrunk region is not sufficiently small, we divide it into two regions and next contract each of them. On the basis of theoretical considerations and numerical experiments the division procedure has been elaborated as follows. We select these sides of the hyperrectangle which either include the point $v_i = w = 0.25$ or are located to the right of this point and choose the largest of them. If the chosen side contains the point $v_i = w$, then this point is used for the division. Otherwise, the division is executed at the middle point of this side. The process is continued until a region having the largest size for $v_i > w$ smaller than 10^{-3}V is obtained. We identify such sizes as points and check whether the region contains a solution to equation (1). During this process some regions containing no solutions are identified and discarded as described in Section II.

A crucial point of this approach is the contraction procedure. An efficient contraction method is proposed in Section II. Section III offers a method enabling us to find the preliminary bounds on the location of all the solutions. Two numerical examples are given in Section IV. The conclusion is formulated in Section V.

II. A CONTRACTION METHOD

Our objective is to develop an efficient contraction method. For this purpose we take into account the matrix equation (1),

consisting of n scalar equations having the form

$$\hat{f}_i(v_i) + \sum_{j=1}^n a_{ij}v_j = b_i, \quad i = 1, \dots, n. \quad (3)$$

Consider a hyperrectangular region $[\mathbf{l}, \mathbf{u}] = [l_1, u_1] \times \dots \times [l_n, u_n]$, ($l_i \leq v_i \leq u_i, i = 1, \dots, n$) and select a subset of the set of equations (3) for which $u_i \leq w$. Without any loss of generality we assume that this subset consists of the first m equations of the set (3). Consequently, the following equations $\hat{f}_1(v_1) = \Delta_1 v_1, \dots, \hat{f}_m(v_m) = \Delta_m v_m$ hold. In such a case equation (1) can be rearranged to give

$$(\mathbf{A}_1 + \mathbf{\Delta})\mathbf{v}_1 + \mathbf{A}_2\mathbf{v}_2 = \mathbf{b}_1,$$

$$\hat{f}_2(\mathbf{v}_2) + \mathbf{A}_3\mathbf{v}_1 + \mathbf{A}_4\mathbf{v}_2 = \mathbf{b}_2, \quad (4)$$

where $\mathbf{v}_1 = [v_1 \dots v_m]^T, \mathbf{v}_2 = [v_{m+1} \dots v_n]^T, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ are submatrices of $\mathbf{A}, \mathbf{\Delta} = \text{diag}(\Delta_1, \dots, \Delta_m), \mathbf{b}_1 = [b_1 \dots b_m]^T, \mathbf{b}_2 = [b_{m+1} \dots b_n]^T, \hat{f}_2(\mathbf{v}_2) = [\hat{f}_{m+1}(v_{m+1}) \dots \hat{f}_n(v_n)]^T$.

By pivoting in succession on the diagonal elements of matrix $(\mathbf{A}_1 + \mathbf{\Delta})$ we rearrange equations (4) to

$$\mathbf{v}_1 + \hat{\mathbf{A}}_2\mathbf{v}_2 = \hat{\mathbf{b}}_1, \quad (5)$$

$$\hat{f}_2(\mathbf{v}_2) + \hat{\mathbf{A}}_4\mathbf{v}_2 = \hat{\mathbf{b}}_2. \quad (6)$$

Next we frame all the scalar functions, appeared in the equation (6), $\hat{f}_i(v_i)$ for $v_i \in [l_i, u_i], i = m+1, \dots, n$, using two parallel straight lines, similarly as in [8], (see Fig. 1). Having determined all the slopes s_{m+1}, \dots, s_n and all the offsets $c_{m+1}^-, \dots, c_n^-, c_{m+1}^+, \dots, c_n^+$ we from the matrix $\mathbf{M}_4 = \text{diag}(s_{m+1}, \dots, s_n)$ and vectors $\mathbf{c}^- = [c_{m+1}^-, \dots, c_n^-]^T, \mathbf{c}^+ = [c_{m+1}^+, \dots, c_n^+]^T$. Let \mathbf{v}^* be an arbitrary solution of the set of equations (3), belonging to the region $[\mathbf{l}, \mathbf{u}]$. Any component v_i^* of \mathbf{v}^* ($i = m+1, \dots, n$) can be considered as a point that lies on the straight line $y_i = s_i v_i + c_i$, where $c_i^- \leq c_i \leq c_i^+$. Hence, the equation (6) can be replaced by the linear equation

$$\mathbf{M}_4\mathbf{v}_2 + \mathbf{c} + \hat{\mathbf{A}}_4\mathbf{v}_2 - \hat{\mathbf{b}}_2 = \mathbf{0}, \quad (7)$$

where components c_i ($i = m+1, \dots, n$) of the vector $\mathbf{c} = [c_{m+1}, \dots, c_n]^T$ are unknown, but the bounds c_i^-, c_i^+ on them are given. We replace the equation (6) by the equation (7) finding

$$\begin{bmatrix} \mathbf{1} & \hat{\mathbf{A}}_2 \\ \mathbf{0} & \mathbf{M}_4 + \hat{\mathbf{A}}_4 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{b}}_1 \\ \hat{\mathbf{b}}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{c} \end{bmatrix}, \quad (8)$$

where $\mathbf{c}^- \leq \mathbf{c} \leq \mathbf{c}^+$. Since

$$\begin{bmatrix} \mathbf{1} & \hat{\mathbf{A}}_2 \\ \mathbf{0} & \mathbf{M}_4 + \hat{\mathbf{A}}_4 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{B} \\ \mathbf{0} & \mathbf{K} \end{bmatrix}, \quad (9)$$

where $\mathbf{B} = -\hat{\mathbf{A}}_2(\mathbf{M}_4 + \hat{\mathbf{A}}_4)^{-1}, \mathbf{K} = (\mathbf{M}_4 + \hat{\mathbf{A}}_4)^{-1}$, then

$$\mathbf{v} = \mathbf{k} - \mathbf{H}\mathbf{c}, \quad (10)$$

where $\mathbf{H} = \begin{bmatrix} \mathbf{B} \\ \mathbf{K} \end{bmatrix} = [h_{ij}]_{n \times (n-m)}, \mathbf{v} = [v_1^T v_2^T]^T = [v_1 \dots v_n]^T, \mathbf{k} = [k_1 \dots k_n]^T = \begin{bmatrix} \hat{\mathbf{b}}_1 + \mathbf{B}\hat{\mathbf{b}}_2 \\ \mathbf{K}\hat{\mathbf{b}}_2 \end{bmatrix}$. Using

the equation (10) we find new bounds on \mathbf{v}^* , such that $p_i^- \leq v_i^* \leq p_i^+$, $i = 1, \dots, n$, where

$$p_i^- = k_i - \sum_{j=1}^{n-m} h_{ij} \varphi_j, \quad \varphi_j = \begin{cases} c_{m+j}^- & \text{if } h_{ij} \leq 0 \\ c_{m+j}^+ & \text{if } h_{ij} > 0 \end{cases},$$

$$p_i^+ = k_i - \sum_{j=1}^{n-m} h_{ij} \delta_j, \quad \delta_j = \begin{cases} c_{m+j}^+ & \text{if } h_{ij} \leq 0 \\ c_{m+j}^- & \text{if } h_{ij} > 0 \end{cases}.$$

Next we define $l_i^1 = \max\{l_i, p_i^-\}$, $u_i^1 = \min\{u_i, p_i^+\}$ and form the region $[\mathbf{l}^1, \mathbf{u}^1] \subset [\mathbf{l}, \mathbf{u}]$, ($\mathbf{l}^1 = [l_1^1 \dots l_n^1]^T$, $\mathbf{u}^1 = [u_1^1 \dots u_n^1]^T$) containing the same solutions as $[\mathbf{l}, \mathbf{u}]$. Next we can use the determined bounds on \mathbf{v}_2 to further improve the bounds on \mathbf{v}_1 applying in a similar way equation (5). The contraction is continued, leading to regions $[\mathbf{l}^2, \mathbf{u}^2]$, $[\mathbf{l}^3, \mathbf{u}^3], \dots$.

If at any stage of the contraction process the lower bound overlaps the upper bound, for at least one component, i.e. $l_k^\mu > u_k^\mu$ for some μ and k , then we conclude that the region $[\mathbf{l}, \mathbf{u}]$ contains no solution.

Note that the method requires the inversion of the matrix $(\mathbf{M}_4 + \hat{\mathbf{A}}_4)$ having the degree $r \times r$ where $r = n - m$ is smaller than n . We repeat this procedure as long as the region is noticeably contracted. If at any step an overlapping occurs, the region is discarded. The more overlappings, the more efficient is the method.

The method can be further improved as follows. When a single region is contracted, usually only some elements of the $r \times r$ diagonal matrix \mathbf{M}_4 change going from one step to another. Frequently it occurs that only a single diagonal element changes. To compute the inversion $(\mathbf{M}_4 + \hat{\mathbf{A}}_4)^{-1}$ efficiently, we apply the Woodbury formula [24], as it is mentioned in [22].

Let us consider an $r \times r$ matrix \mathbf{Q} having given inversion $\mathbf{Q}^{-1} = [\mathbf{Q}_{ij}]_{r \times r}$ and a diagonal $r \times r$ matrix \mathbf{D} with a single unequal to zero element d_k i.e., $\mathbf{D} = \text{diag}(0, \dots, 0, d_k, 0, \dots, 0)$. We express matrix \mathbf{D} in the form $\mathbf{D} = \mathbf{R}\mathbf{S}^{-1}\mathbf{V}$, where $\mathbf{R} = [0 \dots 0 \mathbf{1}_k 0 \dots 0]^T$, $\mathbf{S}^{-1} = d_k$, $\mathbf{V} = \mathbf{R}^T$ and apply the Woodbury formula [24]

$$(\mathbf{Q} + \mathbf{R}\mathbf{S}^{-1}\mathbf{V})^{-1} = \mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{R}(\mathbf{S} + \mathbf{V}\mathbf{Q}^{-1}\mathbf{R})^{-1}\mathbf{V}\mathbf{Q}^{-1}. \quad (11)$$

Hence, we obtain

$$(\mathbf{Q} + \mathbf{D})^{-1} = \mathbf{Q}^{-1} - \begin{bmatrix} \mathbf{Q}_{1k} \\ \vdots \\ \mathbf{Q}_{rk} \end{bmatrix} (d_k^{-1} + \mathbf{Q}_{kk})^{-1} [\mathbf{Q}_{k1} \dots \mathbf{Q}_{kr}]. \quad (12)$$

Equation (12) significantly improves computing the inversion of matrix $(\mathbf{Q} + \mathbf{D})$.

Generally, the matrix \mathbf{D} contains $\gamma \leq r$ unequal to zero diagonal elements located at the crossing of $k_1, k_2, \dots, k_\gamma$ rows and columns. In such a case the Woodbury formula leads to the equation

$$(\mathbf{Q} + \mathbf{D})^{-1} = \mathbf{Q}^{-1} - \begin{bmatrix} \mathbf{Q}_{1k_1} & \dots & \mathbf{Q}_{1k_\gamma} \\ \vdots & \ddots & \vdots \\ \mathbf{Q}_{rk_1} & \dots & \mathbf{Q}_{rk_\gamma} \end{bmatrix} \times$$

$$\times \left[\left(\begin{array}{ccc} d_{k_1}^{-1} & & \\ & \ddots & \\ & & d_{k_\gamma}^{-1} \end{array} \right) + \begin{bmatrix} \mathbf{Q}_{k_1 k_1} & \dots & \mathbf{Q}_{k_1 k_\gamma} \\ \vdots & \ddots & \vdots \\ \mathbf{Q}_{k_\gamma k_1} & \dots & \mathbf{Q}_{k_\gamma k_\gamma} \end{bmatrix} \right]^{-1} \times \begin{bmatrix} \mathbf{Q}_{k_1 1} & \dots & \mathbf{Q}_{k_1 r} \\ \vdots & \ddots & \vdots \\ \mathbf{Q}_{k_\gamma 1} & \dots & \mathbf{Q}_{k_\gamma r} \end{bmatrix} \quad (13)$$

Using a direct method of inversion of an $r \times r$ matrix we execute r^3 multiplications. Applying the Woodbury formula (13), we need γ^3 multiplications for finding the inversion of the matrix having the size $\gamma \times \gamma$, $\gamma r(r + \gamma)$ multiplications needed for finding the product of the matrices which appear on the right hand side of equation (13), and γ inversions of d_{k_i} ($i = 1, \dots, \gamma$). To determine γ so that the total number of multiplications required by the direct method is the same as using the Woodbury formula, we form the equation

$$r^3 - \gamma^3 - \gamma r(r + \gamma) - \gamma = 0$$

and substitute $\gamma = \xi r$. Hence, there follows the equation $\xi^3 + \xi^2 + \xi - 1 = 0$, where the small term $\frac{1}{r^3}\gamma$ has been neglected. The equation has the solution $\xi = 0.543$. Thus, if $\gamma < 0.543r$, the Woodbury formula requires a smaller number of multiplications than the direct method. In the implementation of this approach, the Woodbury formula is applied when the number of diagonal elements of the matrix \mathbf{M}_4 , which are changed going from one step to another, is less than or equal to $0.5r$.

Thus, the method developed in this section requires finding inversions of matrices having degree $r = n - m$, smaller than n , and improves computing these inversions using the above described procedure, based on Woodbury's formula. Numerical examples show that it speeds up the computations 20-25%.

The contraction method, developed in this section, has been applied in the procedure of successive contraction, division and elimination as described in Section I. The algorithm operating in this way will be called the **algorithm A**.

III. FINDING A REGION CONTAINING ALL THE SOLUTIONS

The **algorithm A** enables us to find all the DC solutions in the hyperrectangular region $[\tilde{\mathbf{l}}, \tilde{\mathbf{u}}]$. The efficiency of this algorithm would be considerably improved if the preliminary bounds on the location of all the solutions, inside the region $[\tilde{\mathbf{l}}, \tilde{\mathbf{u}}]$, could be found quickly and easily. The method developed in this section accomplishes this aim. It is based on the idea of the monotonic function [26] and the mathematical backgrounds briefly described below.

A. Mathematical Backgrounds

Definition 1

Let us consider a function $\mathbf{g}(\mathbf{x})$ mapping R^n onto R^n . Function $\mathbf{g}(\mathbf{x})$ is called a monotonic function when $\mathbf{g}(\mathbf{x}_1) \leq \mathbf{g}(\mathbf{x}_2)$ implies that $\mathbf{x}_1 \leq \mathbf{x}_2$ for arbitrary \mathbf{x}_1 and \mathbf{x}_2 , where the inequalities hold for all components.

Definition 2

Let us consider a function $f(x)$ mapping R^1 onto R^1 . Function $f(x)$ is called a strictly monotonically increasing function if $f(x_1) < f(x_2)$ whenever, $x_1 < x_2$, for arbitrary x_1 and x_2 .

The following lemma will be proved.

Lemma 1

Let $\mathbf{f}(\mathbf{x}) = [f_1(x_1) \dots f_n(x_n)]^T$ be a vector function with $f_i(x_i)$ being a strictly monotonically increasing function mapping R^1 onto R^1 ($i = 1, \dots, n$) and $\mathbf{C} = [c_{ij}]_{n \times n}$ be the matrix that satisfies the conditions:

$$c_{ii} > 0, \quad c_{ij} \leq 0 \quad \text{for} \quad i \neq j, \quad i, j = 1, \dots, n, \quad (14)$$

$$\sum_{j=1}^n c_{ij} \geq 0, \quad i = 1, \dots, n. \quad (15)$$

Then the function

$$\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \mathbf{C}\mathbf{x} \quad (16)$$

is a monotonic function.

Proof

The matrix \mathbf{C} that satisfies the conditions (14) and (15) belongs to the class P_0 ([25] Theorem 7). Hence, according to the Sandberg-Willson theorem ([25] Theorem 3), the equation

$$\mathbf{g}(\mathbf{x}) = \mathbf{d} \quad (17)$$

has a unique solution for each $\mathbf{d} \in R^n$.

For arbitrary n -vectors \mathbf{d}_1 and \mathbf{d}_2 such that $\mathbf{d}_1 \leq \mathbf{d}_2$, it holds

$$\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2) + \mathbf{C}(\mathbf{x}_1 - \mathbf{x}_2) \leq \mathbf{0}, \quad (18)$$

where \mathbf{x}_1 (\mathbf{x}_2) is the unique solution of the equation $\mathbf{g}(\mathbf{x}) = \mathbf{d}_1$ ($\mathbf{g}(\mathbf{x}) = \mathbf{d}_2$). Since $f_i(x_i)$, $i = 1, \dots, n$, are strictly monotonically increasing functions, then

$$\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2) = \mathbf{M}(\mathbf{x}_1 - \mathbf{x}_2), \quad (19)$$

where $\mathbf{M} = \text{diag}(m_1, \dots, m_n)$, $m_i > 0$, $i = 1, \dots, n$. Combining (18) and (19) yields

$$\mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) \leq \mathbf{0}, \quad (20)$$

where $\mathbf{K} = [k_{ij}]_{n \times n} = (\mathbf{M} + \mathbf{C})$, $k_{ii} > 0$, $k_{ij} \leq 0$ for $i \neq j$, $i, j = 1, \dots, n$, $\sum_{j=1}^n k_{ij} > 0$, $i = 1, \dots, n$. According to the

Collatz theorem [26], p. 45, the matrix \mathbf{K} is monotonic (and in particular $\det \mathbf{K} \neq 0$), hence, we obtain $\mathbf{x}_1 - \mathbf{x}_2 \leq \mathbf{0}$.

B. Preliminaries

Since we are interested in the solutions belonging to the region $[\tilde{\mathbf{l}}, \tilde{\mathbf{u}}]$, the function $\hat{\mathbf{f}}(\mathbf{v})$ can be changed outside this region. Let the modified function be as follows

$$f_i(v_i) = \begin{cases} \hat{f}_i(v_i) & \text{for } \tilde{l}_i \leq v_i \leq \tilde{u}_i \\ \hat{f}_i(\tilde{u}_i) + (\hat{f}'_i(\tilde{u}_i) + \gamma_i)(v_i - \tilde{u}_i) & \text{for } v_i > \tilde{u}_i \\ \hat{f}_i(\tilde{l}_i) + (\hat{f}'_i(\tilde{l}_i) + \eta_i)(v_i - \tilde{l}_i) & \text{for } v_i < \tilde{l}_i \end{cases} \quad (21)$$

$i = 1, \dots, n,$

where γ_i and η_i are nonnegative constants chosen in the way described at the end of this section. Each function $f_i(v_i)$ is strictly monotonically increasing, mapping R^1 onto R^1 .

Let $\mathbf{E} = [e_{ij}]_{n \times n}$ be a matrix obtained from the matrix \mathbf{A} by replacing all its positive off-diagonal elements by zeros. Thus, the matrix \mathbf{E} satisfies the following conditions

$$e_{ii} = a_{ii} > 0, \quad i = 1, \dots, n, \quad (22)$$

$$e_{ij} \leq 0, \quad i \neq j, \quad i, j = 1, \dots, n. \quad (23)$$

The matrix composed of the positive off-diagonal elements of the matrix \mathbf{A} and the other elements set to zero will be labeled \mathbf{P} . Hence, the matrix \mathbf{P} is given by

$$\mathbf{P} = \mathbf{A} - \mathbf{E}. \quad (24)$$

For the matrix \mathbf{E} we compute

$$\beta_i = \sum_{j=1}^n e_{ij} \quad i = 1, \dots, n \quad (25)$$

and define

$$\partial_i = \begin{cases} -\beta_i & \text{if } \beta_i \leq 0 \\ 0 & \text{if } \beta_i > 0. \end{cases} \quad (26)$$

Next we form matrices $\mathbf{C} = [c_{ij}]_{n \times n}$ and $\mathbf{N} = [n_{ij}]_{n \times n}$, as follows:

$$\mathbf{C} = \mathbf{E} + \text{diag}(\partial_1, \dots, \partial_n), \quad (27)$$

$$\mathbf{N} = -\text{diag}(\partial_1, \dots, \partial_n). \quad (28)$$

The matrix \mathbf{C} satisfies the conditions (14) and (15).

\mathbf{N} is a diagonal matrix having nonpositive diagonal elements $(-\partial_i)$, $i = 1, \dots, n$. As a result the matrix \mathbf{A} is decomposed into three matrices

$$\mathbf{A} = \mathbf{C} + \mathbf{N} + \mathbf{P}. \quad (29)$$

Next we write the equation

$$\mathbf{f}(\mathbf{v}) + (\mathbf{C} + \mathbf{N} + \mathbf{P})\mathbf{v} - \mathbf{b} = \mathbf{0}, \quad (30)$$

which has the same solutions, in the region $[\tilde{\mathbf{l}}, \tilde{\mathbf{u}}]$, as the equation (1).

Let us consider the function

$$\mathbf{g}(\mathbf{x}) - \mathbf{w}(\mathbf{y}, \mathbf{z}), \quad (31)$$

where $\mathbf{x} = [x_1 \dots x_n]^T$, $\mathbf{y} = [y_1 \dots y_n]^T$, $\mathbf{z} = [z_1 \dots z_n]^T$ are elements of R^n ,

$$\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \mathbf{C}\mathbf{x} \quad (32)$$

is a monotonic function

$$\mathbf{w}(\mathbf{y}, \mathbf{z}) = \mathbf{b} - \mathbf{N}\mathbf{y} - \mathbf{P}\mathbf{z}, \quad (33)$$

where \mathbf{b} is the vector which appears in equation (30), \mathbf{N} is the diagonal matrix (28) with nonpositive diagonal elements, \mathbf{P} is the matrix (24) having all nonnegative elements.

C. Contraction of the Bounds on the Solutions

Let \mathbf{v}^* be an arbitrary solution of the equation (30), then

$$\mathbf{g}(\mathbf{v}^*) - \mathbf{w}(\mathbf{v}^*, \mathbf{v}^*) = \mathbf{f}(\mathbf{v}^*) + (\mathbf{C} + \mathbf{N} + \mathbf{P})\mathbf{v}^* - \mathbf{b} = \mathbf{0}, \quad (34)$$

or

$$\mathbf{g}(\mathbf{v}^*) = \mathbf{w}(\mathbf{v}^*, \mathbf{v}^*). \quad (35)$$

If there exist $\mathbf{l}^{(0)} \in R^n$ and $\mathbf{u}^{(0)} \in R^n$, such that

$$\mathbf{l}^{(0)} \leq \tilde{\mathbf{l}}, \quad \mathbf{u}^{(0)} \geq \tilde{\mathbf{u}}, \quad (36)$$

$$\mathbf{g}(\mathbf{l}^{(0)}) \leq \mathbf{w}(\mathbf{l}^{(0)}, \mathbf{u}^{(0)}), \quad (37)$$

$$\mathbf{g}(\mathbf{u}^{(0)}) \geq \mathbf{w}(\mathbf{u}^{(0)}, \mathbf{l}^{(0)}), \quad (38)$$

then for any solution $\mathbf{v}^* \in [\mathbf{l}^{(0)}, \mathbf{u}^{(0)}]$ of the equation (30) we may write

$$\mathbf{w}(\mathbf{l}^{(0)}, \mathbf{u}^{(0)}) \leq \mathbf{w}(\mathbf{v}^*, \mathbf{v}^*) \leq \mathbf{w}(\mathbf{u}^{(0)}, \mathbf{l}^{(0)}). \quad (39)$$

Since \mathbf{g} is a monotonic function (see Lemma 1), the relation (39) implies that

$$\begin{aligned} \mathbf{l}^{(1)} &= \mathbf{g}^{-1}(\mathbf{w}(\mathbf{l}^{(0)}, \mathbf{u}^{(0)})) \leq \mathbf{g}^{-1}(\mathbf{w}(\mathbf{v}^*, \mathbf{v}^*)) \leq \\ &\leq \mathbf{g}^{-1}(\mathbf{w}(\mathbf{u}^{(0)}, \mathbf{l}^{(0)})) = \mathbf{u}^{(1)}. \end{aligned} \quad (40)$$

On the basis of (35) we get

$$\mathbf{g}^{-1}(\mathbf{w}(\mathbf{v}^*, \mathbf{v}^*)) = \mathbf{v}^*. \quad (41)$$

Substituting (41) into (39) and taking into account (40) yields

$$\mathbf{l}^{(1)} \leq \mathbf{v}^* \leq \mathbf{u}^{(1)}. \quad (42)$$

Since \mathbf{v}^* is an arbitrary solution belonging to the region $[\mathbf{l}^{(0)}, \mathbf{u}^{(0)}]$, the relation (42) states that the region $[\mathbf{l}^{(1)}, \mathbf{u}^{(1)}]$ contains all the solutions belonging to the region $[\mathbf{l}^{(0)}, \mathbf{u}^{(0)}]$. Using (40) we may write

$$\mathbf{g}(\mathbf{l}^{(1)}) = \mathbf{w}(\mathbf{l}^{(0)}, \mathbf{u}^{(0)}). \quad (43)$$

Since \mathbf{g} is a monotonic function then equation (43) and inequality (37) imply that

$$\mathbf{l}^{(1)} \geq \mathbf{l}^{(0)}. \quad (44)$$

Similarly, using the equation

$$\mathbf{g}(\mathbf{u}^{(1)}) = \mathbf{w}(\mathbf{u}^{(0)}, \mathbf{l}^{(0)}). \quad (45)$$

we prove that

$$\mathbf{u}^{(1)} \leq \mathbf{u}^{(0)}. \quad (46)$$

Furthermore, $\mathbf{g}(\mathbf{l}^{(1)}) \leq \mathbf{w}(\mathbf{l}^{(1)}, \mathbf{u}^{(1)})$, $\mathbf{g}(\mathbf{u}^{(1)}) \geq \mathbf{w}(\mathbf{u}^{(1)}, \mathbf{l}^{(1)})$. Continuing this process, the sequences $\{\mathbf{l}^{(j)}\}$ and $\{\mathbf{u}^{(j)}\}$ are generated, where

$$\begin{aligned} \mathbf{l}^{(j)} &= \mathbf{g}^{-1}(\mathbf{w}(\mathbf{l}^{(j-1)}, \mathbf{u}^{(j-1)})), \\ \mathbf{u}^{(j)} &= \mathbf{g}^{-1}(\mathbf{w}(\mathbf{u}^{(j-1)}, \mathbf{l}^{(j-1)})), \\ \mathbf{l}^{(j)} &\leq \mathbf{v}^* \leq \mathbf{u}^{(j)}. \end{aligned} \quad (47)$$

The bounds $\mathbf{l}^{(j)}$ and $\mathbf{u}^{(j)}$ are determined solving the equations $\mathbf{g}(\mathbf{l}^{(j)}) = \mathbf{w}(\mathbf{l}^{(j-1)}, \mathbf{u}^{(j-1)})$ and $\mathbf{g}(\mathbf{u}^{(j)}) = \mathbf{w}(\mathbf{u}^{(j-1)}, \mathbf{l}^{(j-1)})$ using the Newton-Raphson algorithm. The sequence $\{\mathbf{l}^{(j)}\}$ is monotonically increasing and bounded

from above, whereas $\{\mathbf{u}^{(j)}\}$ is monotonically decreasing and bounded from below. Consequently, both sequences are convergent, $\lim_{j \rightarrow \infty} \mathbf{l}^{(j)} = \mathbf{l}^*$, $\lim_{j \rightarrow \infty} \mathbf{u}^{(j)} = \mathbf{u}^*$. Furthermore, taking into account (47) we may write

$$\mathbf{l}^* \leq \mathbf{v}^* \leq \mathbf{u}^*. \quad (48)$$

Thus, we obtain region $[\mathbf{l}^*, \mathbf{u}^*] \subset [\mathbf{l}^{(j)}, \mathbf{u}^{(j)}]$ containing all the solutions belonging to $[\tilde{\mathbf{l}}, \tilde{\mathbf{u}}]$. Hence, the region $[\mathbf{l}^{new}, \mathbf{u}^{new}]$ where $\mathbf{l}^{new} = [l_1^{new}, \dots, l_n^{new}]^T$, $\mathbf{u}^{new} = [u_1^{new}, \dots, u_n^{new}]^T$ and $l_i^{new} = \max\{\tilde{l}_i, l_i^*\}$, $u_i^{new} = \min\{\tilde{u}_i, u_i^*\}$, $i = 1, \dots, n$ is a contracted region containing all the solutions belonging to $[\tilde{\mathbf{l}}, \tilde{\mathbf{u}}]$.

A crucial point of this approach is finding of the initial bounds $\mathbf{l}^{(0)}$, $\mathbf{u}^{(0)}$ on the solutions, satisfying the constraints (36)–(38). Below, it is shown that for arbitrary $\mathbf{l}^{(0)} < \tilde{\mathbf{l}}$ and $\mathbf{u}^{(0)} > \tilde{\mathbf{u}}$ the restrictions (37)–(38) can be always satisfied choosing appropriate constants γ_i and η_i .

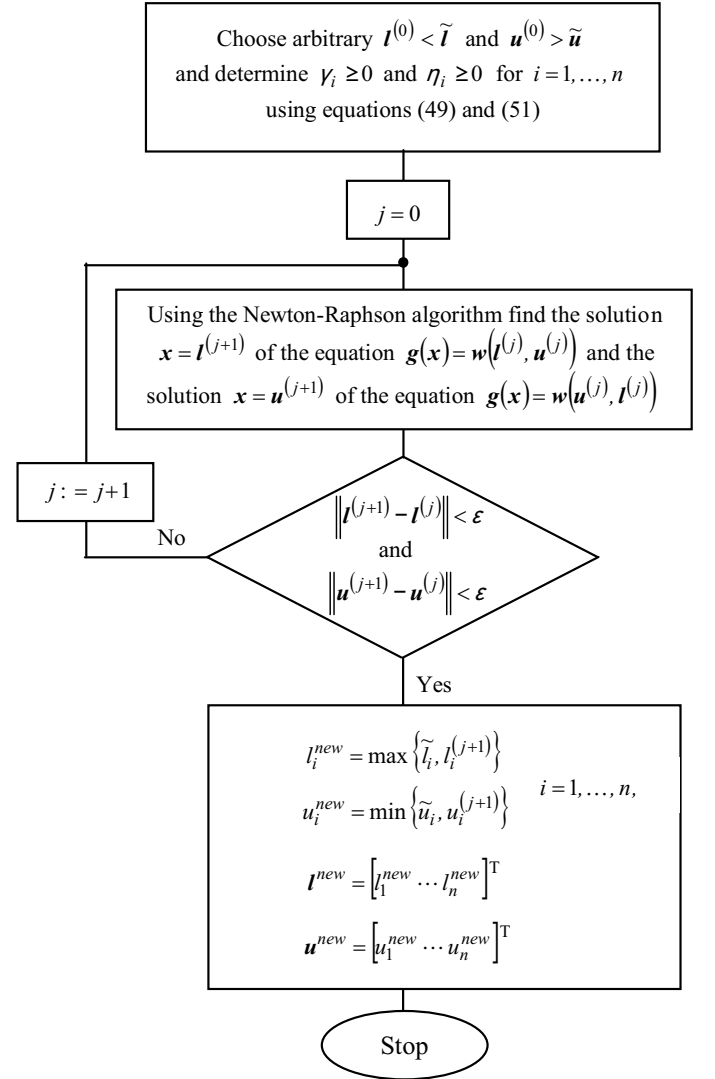


Fig. 2. Flowchart of the method for finding the bounds on the solutions of equation (1).

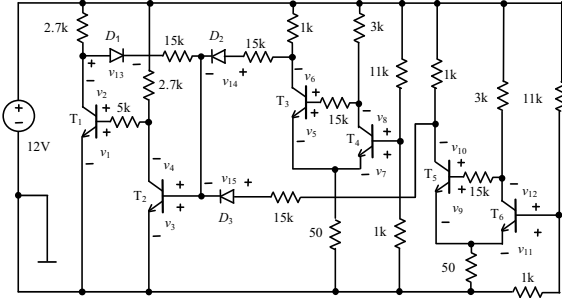


Fig. 3. The circuit for Example 1.

D. Procedure for Finding γ_i and η_i

To satisfy the conditions (37)–(38) we can choose arbitrarily $\mathbf{l}^{(0)} < \tilde{\mathbf{l}}$ and $\mathbf{u}^{(0)} > \tilde{\mathbf{u}}$ and next select γ_i and η_i (see equation (21)) as follows:

$$\gamma_i = \begin{cases} \hat{\gamma}_i & \text{if } \hat{\gamma}_i > 0 \\ 0 & \text{if } \hat{\gamma}_i \leq 0, \end{cases} \quad (49)$$

where

$$\hat{\gamma}_i = \frac{1}{u_i^{(0)} - \tilde{u}_i} \left(b_i - \hat{f}_i(\tilde{u}_i) - \hat{f}'_i(\tilde{u}_i)(u_i^{(0)} - \tilde{u}_i) + \right. \\ \left. - \sum_{j=1}^n (c_{ij} + n_{ij})u_j^{(0)} - \sum_{j=1}^n p_{ij}l_j^{(0)} \right), \quad (50)$$

$$\eta_i = \begin{cases} \hat{\eta}_i & \text{if } \hat{\eta}_i > 0 \\ 0 & \text{if } \hat{\eta}_i \leq 0, \end{cases} \quad (51)$$

where

$$\hat{\eta}_i = \frac{1}{\tilde{l}_i - l_i^{(0)}} \left(\hat{f}_i(\tilde{l}_i) + \hat{f}'_i(\tilde{l}_i)(l_i^{(0)} - \tilde{l}_i) + \right. \\ \left. + \sum_{j=1}^n (c_{ij} + n_{ij})l_j^{(0)} + \sum_{j=1}^n p_{ij}u_j^{(0)} - b_i \right). \quad (52)$$

Thus, γ_i and η_i depend on arbitrarily chosen $\mathbf{l}^{(0)} < \tilde{\mathbf{l}}$ and $\mathbf{u}^{(0)} > \tilde{\mathbf{u}}$. Numerical experiments show that the choice $\mathbf{l}^{(0)}$ and $\mathbf{u}^{(0)}$ has little influence on the convergence of the contraction procedure.

The flowchart of this method is shown in Fig. 2.

To illustrate the efficiency of the method for finding the bounds on the location of all the solutions, developed in this section, we consider a numerical example.

Example 1

Let us consider the benchmark circuit (e.g. [8], [11]) shown in Fig. 3. The diodes and transistors are characterized by the Ebers-Moll model with the following parameters: $\alpha_F = 0.99$, $\alpha_R = 0.5$, $K_i = 10.10\text{fA}$ ($i = 1, 3, \dots, 11$), $K_i = 20.00\text{fA}$ ($i = 2, 4, \dots, 12$), $K_i = 10.00\text{fA}$ ($i = 13, \dots, 15$), $\lambda = 38.6635\text{V}^{-1}$.

Using the method developed in this section, with $E = 12\text{V}$, $I_i = 100\text{mA}$ ($i = 1, \dots, 15$), we find the bounds on the

TABLE I
BOUNDS ON THE SOLUTIONS

	Lower bounds on the solutions provided by the contraction method [V]	Upper bounds on the solutions provided by the contraction method [V]	Lower bounds determined by the computed solutions [V]	Upper bounds determined by the computed solutions [V]
v_1	-4.387	0.712	0.038	0.706
v_2	-12.000	0.686	-10.337	0.664
v_3	-4.731	0.716	0.408	0.709
v_4	-12.000	0.691	-7.631	0.671
v_5	-1.367	0.721	0.103	0.720
v_6	-12.000	0.649	-11.031	0.636
v_7	-0.176	0.693	0.404	0.691
v_8	-12.000	0.656	-9.219	0.589
v_9	-1.367	0.721	0.103	0.720
v_{10}	-12.000	0.649	-11.031	0.636
v_{11}	-0.176	0.693	0.404	0.691
v_{12}	-12.000	0.656	-9.219	0.589
v_{13}	-5.789	0.658	-0.366	0.642
v_{14}	-2.103	0.658	-0.025	0.645
v_{15}	-2.103	0.658	-0.025	0.645

solutions, shown in the first and second columns of Table 1. This circuit has 11 DC solutions, bounded by the numbers shown in the third and fourth columns of Table I. The bounds obtained by the proposed method very well frame the solutions, which confirms efficiency of the method.

The method for finding the bounds on the location of all the solutions, proposed in this section, has been merged into the **algorithm A** as the preliminary procedure. The algorithm formed in this way will be called the **algorithm B**.

IV. NUMERICAL EXAMPLES

The **algorithms A** and **B** have been implemented in Delphi and tested on a set of 25 circuits. The computations have been executed using PC Pentium Core 2 Duo E6400. For the Example 1 the time consumed by the **algorithm B** is 15ms, whereas the time consumed by the **algorithm A** is 94ms. Two other examples are given underneath.

Example 2

Let us consider the circuit containing 15 transistors, shown in Fig. 4. The transistors are characterized by the Ebers-Moll model with the following parameters: $\alpha_F = 0.99$, $\alpha_R = 0.5$, $K_i = 7.1202\text{fA}$ ($i = 1, 3, \dots, 29$), $K_i = 14.0980\text{fA}$ ($i = 2, 4, \dots, 30$), $R_E = R_C = R_B = 0$, $\lambda = 38.6635\text{V}^{-1}$. To find all the DC solutions we apply the **algorithm B** with $E = 6.5\text{V}$, $I_i = 80\text{mA}$ ($i = 1, \dots, 30$) finding 3 solutions: $\mathbf{v}^{(1)} = [0.688 \ -0.629 \ 0.629 \ -3.534 \ 0.688 \ 0.006 \ 0.688 \ -2.788 \ 0.198 \ -1.123 \ 0.688 \ -0.000 \ 0.636 \ -3.333 \ 0.701 \ -0.041 \ 0.696 \ 0.689 \ 0.725 \ 0.427 \ 0.680 \ -0.000 \ -0.689 \ 0.000 \ 0.673 \ -0.000 \ -1.432 \ -4.350 \ 0.678 \ -0.835]^T$ $\mathbf{v}^{(2)} = [0.688 \ -0.629 \ 0.629 \ -3.538 \ 0.688 \ -0.011 \ 0.671 \ -2.810 \ 0.668 \ -0.546 \ 0.671 \ -0.000 \ 0.647 \ -2.839 \ 0.672 \ -1.197 \ 0.657 \ -0.125 \ 0.708 \ -0.539 \ 0.654 \ -0.000 \ 0.125 \ -0.000 \ 0.684 \ -0.000 \ -0.490 \ -3.409 \ 0.678 \ -0.835]^T$ $\mathbf{v}^{(3)} = [0.688 \ -0.629 \ 0.629 \ -3.540 \ 0.688 \ -0.253 \ 0.429 \ -2.960 \ 0.690 \ 0.624 \ 0.525 \ -0.000 \ 0.659 \ -2.565 \ -0.211 \ -2.290 \ 0.655 \ -0.673 \ 0.690 \ -1.193 \ 0.532 \ -0.000 \ 0.672 \ -0.000 \ 0.688 \ -0.000 \ 0.141 \ -2.779 \ 0.678 \ -0.835]^T$

algorithm B with $E = 7.5\text{V}$, $I_i = 15\text{mA}$ ($i = 1, \dots, 35$) finding 5 solutions, each represented by 35-dimension vector.

The voltages at the nodes A, B, C, D, E corresponding to these solutions are listed in Table II.

The time consumed by the **algorithm B** is 38.47s. The **algorithm A** gives the same solutions in 56.84s.

It should be stressed that the complexity of all the known methods, which guarantee finding all the DC solutions, dramatically increases as the size of the circuit grows. This is why the benchmark transistor circuits discussed in the world literature are rather simple, described by a set up to 20 equations. The circuit in Example 2 is described by a set of 30 equations and the circuit in Example 3 by a set of 35 equations.

The largest diode-transistor practical circuit which can be efficiently analysed by the proposed in this paper algorithm is described by a set of 50 equations.

V. CONCLUSION

Both methods proposed in this paper lead to a procedure for finding all the DC solutions of diode-transistor circuits. The first method considerably contracts and eliminates many hyperrectangular regions in which the solutions are sought. The other one significantly shrinks the preliminary region containing all the solutions. A lot of numerical examples show that the proposed approach is efficient and enables us to analyse larger circuits.

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